Propagation of the homogeneous wavefront set for Schrödinger equations

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• Some wavefront sets

Definition (S. Nakamura [2]) Let \( u \in \mathcal{S}'(\mathbb{R}^n) \) and \((z_0, \zeta_0) \in \mathbb{R}^{2n} \setminus \{0\}\). Then

\[
(z_0, \zeta_0) \notin \text{HWF}(u) \iff \exists \varphi \in C_0^\infty(\mathbb{R}^{2n}) \text{ s.t. } \varphi(z_0, \zeta_0) \neq 0, \text{ and } \|a^w(hz, hD_z)u\|_{L^2} \leq C_N h^N (\forall N > 0).
\]

Remark If we delete \( h \) in front of \( z \) in the above definition, this gets to be a characterization of the wavefront set.

Definition (J. Wunsch [3]) For an operator \( A = a^w(x, D) \in \text{Op } S(1, \langle z \rangle^{-2}dz^2 + \langle \zeta \rangle^{-2}d\zeta^2) \)

we can consider well-defined characteristic set also for the spatial direction:

\[
\Sigma_{sc}(A) = \{(z, \zeta) \in S^{n-1} \times \mathbb{R}^n; \lim \inf_{t \to +\infty} |a(tz, \zeta)| = 0\} \bigcup \{(z, \zeta) \in \mathbb{R}^n \times S^{n-1}; \lim \inf_{t \to +\infty} |a(z, t\zeta)| = 0\}. \tag{disjoint union}
\]

Then we define the scattering and the quadratic scattering wavefront set by

\[
\text{WF}_{sc}(u) = \bigcap \{\Sigma_{sc}(A); A \in \text{Op } S(1, \langle z \rangle^{-2}dz^2 + \langle \zeta \rangle^{-2}d\zeta^2), Au \in \mathcal{S}(\mathbb{R}^n)\},
\]

\[
\text{WF}_{qsc}(u) = \text{WF}_{sc}(\tilde{u}), \quad \tilde{u}(q) = u((1 + \langle q \rangle)^{-\frac{2}{4}}q).
\]

Remark Since \( \mathcal{F}a^w(z, D_z) \mathcal{F}^{-1} = a^w(-D_z, \zeta) \), we have a correspondence

\[
\text{WF}_{sc}(u) \cap (\mathbb{R}^n \times S^{n-1}) \longleftrightarrow \text{WF}(u), \quad \text{WF}_{sc}(u) \cap (S^{n-1} \times \mathbb{R}^n) \longleftrightarrow \text{WF}(\mathcal{F}u).
\]

Put \( z = (1 + \langle q \rangle)^{-\frac{1}{4}}q \), then \( \langle z \rangle^{-2} = \langle q \rangle^{-1} \). If we embed \( \mathbb{R}^n \) into \( S^n_+ \) through the stereographic projection:

\[
\text{SP}: \mathbb{R}^n \to S^n_+ = \{w \in \mathbb{R}^{n+1}; |w| = 1, w_n \geq 0\}, \quad z \mapsto \left(\frac{z}{\sqrt{1 + |z|^2}}, \frac{1}{\sqrt{1 + |z|^2}}\right),
\]

then the qsc-wf set is the sc-wf set of \( u \) on \( \mathbb{R}^n \subset S^n_+ \) that has a new \( C^\infty \) structure whose boundary defining function is the square of the original one. The qsc-wf set is nothing but the coordinate-changed sc-wf set, so there is a correspondence

\[
\text{WF}_{qsc}(u) \cap (\mathbb{R}^n \times S^{n-1}) \longleftrightarrow \text{WF}_{sc}(u) \cap (\mathbb{R}^n \times S^{n-1}) \longleftrightarrow \text{WF}(u).
\]

For \( \text{WF}_{qsc}(u) \cap (S^{n-1} \times \mathbb{R}^n) \) we have another kind of correspondence.

Theorem 1 Let \( \Psi: \mathbb{R}^{n-1} \setminus \{0\} \to \text{GL}(n, \mathbb{R}), z \mapsto \Psi(z) = (\delta_{ij} + z^i z^j |z|^2)_{ij}, \) then

\[
\{(tz, t\zeta) \in \mathbb{R}^{2n}; (z, \zeta) \in \text{WF}_{qsc}(u) \cap (S^{n-1} \times \mathbb{R}^n), t > 0\} = \{(z, \Psi(z)\zeta) \in \mathbb{R}^{2n}; (z, \zeta) \in \text{HWF}(u) \setminus (\{0\} \times \mathbb{R}^n)\}.
\]
Propagation of singularities

Let $\mathbb{R}^n$, $g$ be a Riemannian manifold and consider the Schrödinger equation:

$$(i\partial_t + \triangle_g - V(z))u_t(z) = 0, \quad \triangle_g = \frac{1}{2} \partial_i g^{ij}(z) \partial_j.$$ 

Suppose that $g$ is of the form

$$g = \frac{dx^2}{x^4} + \frac{h(x, y, dx, dy)}{x^2}, \quad x = |z|^{-1}, \quad y: \text{local coordinates of } S^{n-1}$$

for $z \in \mathbb{R}^n$ far from the origin, where $h$ is a 2-cotensor on $\mathbb{R}^n$ that approaches some Riemannian metric on $S^{n-1}$ as $x \to 0$. In other words, $g$ is asymptotically conic.

$V$ is assumed to be in $C^1(\mathbb{R}^n; \mathbb{R})$ and satisfies for some $\nu < 2$

$$|\partial_z^\alpha u(z)| \leq C\alpha \langle z \rangle^{\nu-|\alpha|} \quad \forall \alpha \in \mathbb{Z}_+^n.$$ 

Under these assumptions the potential term can be completely ignored as a small perturbation to the Laplacian.

**Theorem 2** Let $u_0 \in L^2$, $\omega_- \in S^{n-1}$, and $t_0 > 0$, and assume that $(-t_0 \omega_-, \omega_-) \notin \text{HWF}(u_0)$. Then, if $\gamma(t) = (z(t), \zeta(t))$ is a free backward nontrapped classical trajectory with limiting direction $\omega_-$, i.e., if

$$\gamma(t) = (\partial_t p(\gamma(t)), -\partial_z p(\gamma(t))), \quad p(z, \zeta) = \frac{1}{2} \sum_{i,j=1}^n g^{ij}(z)\zeta_i \zeta_j,$$

$$\lim_{t \to -\infty} |z(t)| = \infty, \quad \omega_- = \lim_{t \to -\infty} \zeta(t)/|\zeta(t)| = -\lim_{t \to -\infty} z(t)/|z(t)|,$$

then we have

$$\text{WF}(u_{t_0}) \cap \{\gamma(t); t \in \mathbb{R}\} = \emptyset.$$ 

**Remark** For our metric, if a trajectory $\gamma$ is backward nontrapped, there always exists a limiting direction $\omega_- \in S^{n-1}$.

Professor S. Nakamura has proved Theorem 2 for asymptotically flat $g$.

A part of results by Wunsch [3] on the Euclidean space with an optimally weak assumption on potential follows from Theorems 1 and 2, since for any $t > 0$ we have the equivalence

$$(-t \omega_-, \omega_-) \in \text{HWF}(u) \iff (-\omega_-, \omega_-/2t) \in \text{WF}_{\text{qsc}}(u).$$

**References**

