Global Existence for Systems of Nonlinear Wave Equations in Exterior Domains

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This talk is based on a joint work with Hideo Kubo (Osaka University). Let $\mathcal{O}(\subset \{x \in \mathbb{R}^3; |x| \leq 1\})$ be either a non-trapping obstacle, or a trapping obstacle which was treated by Ikawa ('82, '88), with smooth boundary.

We set $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$, and consider the Dirichlet problem

$$\begin{align*}
(1) & \quad (\partial_t^2 - c_i^2 \Delta_x) u_i = F_i(u, \partial u), \quad (t, x) \in (0, \infty) \times \Omega, \\
(2) & \quad u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial \Omega, \\
(3) & \quad u(0, x) = \phi(x), \quad (\partial_t u)(0, x) = \psi(x), \quad x \in \Omega,
\end{align*}$$

for $i = 1, \ldots, N$, where $c_i > 0, u = (u_1, \ldots, u_N)$ and $\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla_x)$. In the following, we always suppose that $\phi$ and $\psi$ are small in some suitable norm, and $(\phi, \psi, F)$ satisfies the compatibility condition to infinite order.

First we consider the Cauchy problem with $\Omega = \mathbb{R}^3$. We say that the null condition associated with $(c_1, \ldots, c_N)$ is satisfied if $F_i$ can be written as

$$F_i(u, \partial u) = \sum_{j, k; c_j = c_k = c_i} A_{ijk} Q_0(u_j, u_k; c_i) + \sum_{j, k; c_j = c_k = c_i} \sum_{a, b} B_{ijk}^{ab} Q_{ab}(u_j, u_k)$$

$$+ \sum_{j, k; c_j \neq c_k} \sum_{a, b} C_{ijk}^{ab} (\partial_a u_j)(\partial_b u_k) + \sum_{j, k; c_j \neq c_k} \sum_{a, b} D_{ijk}^{ab} (\partial_a u_j)(\partial_b u_k)$$

$$+ O(|u|^3 + |\partial u|^3) \quad (i = 1, \ldots, N),$$

where the null forms are defined by

$$\begin{align*}
(4) & \quad Q_0(v, w; c) = (\partial_t v)(\partial_t w) - c^2 (\nabla_x v) \cdot (\nabla_x w), \\
(5) & \quad Q_{ab}(v, w) = (\partial_a v)(\partial_b w) - (\partial_b v)(\partial_a w) \quad (0 \leq a < b \leq 3).
\end{align*}$$

This condition was first introduced by Klainerman ('86) for the single speed case $(c_1 = \cdots = c_N = 1)$ and global existence of small solutions under the null condition is proved (see also Christodoulou '86). Klainerman used the vector fields method with

$$S = t\partial_t + x \cdot \nabla_x, \quad L_j = t\partial_j + x_j \partial_t, \quad \Omega_{jk} = x_j \partial_k - x_k \partial_j.$$

This global existence result under the null condition is extended to the multiple speeds case by many authors. $L_j$’s are excluded from the arguments in these works.

Now we consider the Dirichlet problem in the exterior domain $\Omega(\subset \mathbb{R}^3)$. Metcalfe–Nakamura–Sogge ('05) proved

**Theorem 1.** Let $s$ be a sufficiently large integer. Suppose that $F$ satisfies the null condition associated with $(c_1, c_2, \ldots, c_N)$. If $\|\phi\|_{H^{s+2,0}(\Omega)} + \|\psi\|_{H^{s+1,0}(\Omega)} \ll 1$, then the mixed problem (1)–(3) admits a unique solution $u \in C^\infty([0, \infty) \times \overline{\Omega}; \mathbb{R}^N)$.

They used $S$ and $\Omega_{jk}$ as in the Cauchy problem, but because of the boundary, the usage of $S$ makes the argument complicated.

The aim of this talk is to give an alternative approach where $S$ is not used.
We write $\langle a \rangle = \sqrt{1 + |a|^2}$. We define

$$|\varphi(t, x)|_k = \sum_{|\alpha| \leq k} |Z^\alpha \varphi(t, x)|,$$

where $Z = (\Omega_{12}, \Omega_{23}, \Omega_{31}, \partial_t, \partial_1, \partial_2, \partial_3)$. Setting $c_0 = 0$, we also define

$$\Phi_\rho(t, x) = (t - |x|)^\rho \quad \text{for } \rho > 0, \quad \Phi_0(t, x) = \left\{ \log \left( 2 + \frac{(t + |x|)}{(t - |x|)} \right) \right\}^{-1},$$

$$W_{\rho, \kappa}(t, x) = (t + |x|)^\nu \left( \min_{0 \leq j \leq N} \langle c_j t - |x| \rangle \right)^\kappa,$$

where $\chi$ via the cut–off method by Shibata (’83) and Shibata–Tsutsumi (’86). For

Let $Z \geq 0$, we also define

$$N_k[f; W](t) = \sup_{(s, x) \in [0, t] \times \Omega} \langle x \rangle W(s, x) \langle f(s, x) \rangle_k \text{ for a weight function } W,$$

where $\rho, \kappa > 0$.

Theorem 2. Let $v$ be the solution to the Dirichlet problem $(\partial_t^2 - c^2 \Delta_x) v = f$ in $(0, \infty) \times \Omega$, $v(t, x) |_{x \in \partial \Omega} = 0$ for $t \in (0, \infty)$, $(v, \partial_t v) = \bar{v}_0 (\equiv (v_0, v_1))$ at $t = 0$.

Set $r = |x|$, and let $\mu > 0$.

(i) If $(\nu = \rho \geq 1, \kappa > 1)$ or $(\rho \geq 1, \nu = \rho + \mu, \kappa = 1 - \mu)$, then

$$\langle t + r \rangle \Phi_{\rho - 1}(t, x) |v(t, x)|_k \leq C_0 \mathbf{A}_{r+1, k+4} \bar{v}_0 + C \sum_{|\beta| \leq 4} N_k[\partial^\beta f; W_{\nu, \kappa}](t).$$

(ii) If $(\nu > \rho \geq 1, \kappa > 1)$, or $(0 < \rho \leq 1, \nu = 1 + \mu, \kappa = \rho - \mu)$, then we have

$$\langle t + r \rangle \langle r \rangle |\partial v(t, x)|_k \leq C_0 \mathbf{A}_{r+2, k+5} \bar{v}_0 + CN_k[f; W_{\nu, \kappa}](t).$$

(iii) Let $\rho > 0$ and $\kappa > 1$. Then we have

$$\langle r \rangle \langle t + r \rangle |\partial v(t, x)|_k \leq C_0 \mathbf{A}_{r+2, k+5} \bar{v}_0 + CN_k[f; W_{\nu, \kappa}](t).$$

(iv) Let $\rho \leq 2$. If $(\nu = \rho \geq 1, \kappa > 1)$ or $(\rho \geq 1, \nu = \rho + \mu, \kappa = 1 - \mu)$, then

$$\langle r \rangle \langle t + r \rangle \langle r \rangle |\partial v(t, x)|_k \leq C_0 \mathbf{A}_{r+2, k+5} \bar{v}_0 + CN_k[f; W_{\nu, \kappa}](t),$$

where $D_{+c} = \partial_t + c\partial_r$. The above estimates are true for $(t, x) \in [0, \infty) \times \Omega$.

Let $\chi = \chi(x)$ be a cut–off function supported on $|x| \leq 5$. (i)–(iii) are obtained by combining the corresponding estimates for the Cauchy problem (Asakura ’86, Kubota–Yokoyama ’01, K.–Yokoyama ’05) with decay of local character

$$\langle t \rangle \langle \nu(t, \cdot) \rangle_{H^k(\Omega; |\cdot| < 5)} \leq C \left( \|\chi \bar{v}_0\|_{H^{k+1}(\Omega)} + C \sup_{s \in [0, t]} \langle s \rangle \sum_{|\alpha| \leq k} \|\chi \partial^\alpha f(s)\|_{L^2(\Omega)} \right),$$

via the cut–off method by Shibata (’83) and Shibata–Tsutsumi (’86). For $r > 1$, we can obtain (iv) by integrating

$$(\partial_t - c\partial_r) D_{+c}(rZ^\alpha v) = r Z^\alpha f + \frac{c^2}{r} \sum_{1 \leq j < k \leq 3} \Omega_{jk}^2 Z^\alpha v$$

along the certain ray in $\Omega$, using (i) to estimate $\Omega_{jk}^2 Z^\alpha v$. (iv) enables us to treat the null forms without using $S$. 