Sem. on Probab.
Vol.56 1988年
P1-120

SEMINAR ON PROBABILITY

Vol. 56

CONVERGENCE OF MARKOV CHAINS ON A HOMOGENEOUS TREE

Herbert Heyer

1988

確 率 論 セ ミ ナ ー

Sem. on Probab.
Vol.56 1988年
P1-120

CONVERGENCE OF MARKOV CHAINS ON A HOMOGENEOUS TREE

by Herbert Heyer, Tübingen

Contents

Sem. on Probab.
Vol.56 1988年
P1-120

Introduction

The potential theory of Markov chains has been successfully applied by P. Cartier ([2], [3], [4]) to the harmonic analysis of trees. In another direction S. Sawyer in [15] applied harmonic analysis in order to study, in a quantitive way, the transience of random walks on a tree. While Cartier's approach consists of a systematic development of the boundary theory of nonnegative harmonic functions on a tree (including Martin's representation and Fatou's radial convergence), Sawyer relies on the theory of Gelfand pairs and their spherical functions, applied to the given homogeneous tree (including horocycles and the Plancherel measure). In the lectures which are the basis of the present exposition, the author tried to work out the various methodical aspects of the theory with the aim to re-establish the existence of the Martin boundary of a tree and the limit theorem yielding the transience of any isotropic random walk on a homogeneous tree. The proofs of both theorems will be carefully prepared and more or less completely carried out.

We shall briefly report on the organization of the Lecture Notes. In Section 1 we start by presenting some generalities on trees, and show the existence of paths of smallest length (Theorem 1.6) and of geodesics (Theorem 1.14). Section 2 is devoted to the properties of harmonic functions and the Green kernel corresponding to a tree. Here we adopt Cartier's general assumption which in Section 6 will be interpreted as a transience condition for the underlying random walk.

Sem. on Probab.
Vol.56 1988年
P1-120

The main information on the Green kernel is contained in Theorem 2.10. Among the analogues of classical results Theorem 2.12 is the well-known Riesz decomposition. In Section 3 the space of geodesics will be viewed as a boundary of the given tree. We shall discuss the relationship between the set of infinite chains starting at the origin and the set of ends of the tree (Proposition 3.4). Theorem 3.10 provides the existence of the natural compactification of the tree in terms of its boundary. In Section 4 this natural compactification will be interpreted as a Martin compactification (Theorem 4.13). The result is preceded by a study of the Martin kernel (Proposition 4.2) and by the proof of Martin's integral representation of nonnegative harmonic functions (Theorem 4.9). As a first conclusion we will present in Section 5 a probabilistic interpretation of the Martin compactification of a tree. We shall construct a Markov chain on the set of all infinite paths in the tree (Propositions 5.5 and 5.6) and prove that this Markov chain converges for almost all paths towards an element of the Martin boundary (Theorem 5.12). It turns out (Theorem 5.14) that the Martin representing measure is the limiting distribution of the Markov chain. Section 6 is devoted to the discussion of the transience property for certain Markov chains on a tree. It appears (Corollary 6.9) that in the case of a homogeneous tree the canonical Markov chain is always transient. Theorem 6.13 contains all available information obtained under the homogeneity condition for the Martin representing measure. In the subsequent Section 7 a crash report on Gelfand pairs precedes the main Theorem 7.18 in which the transience

Sem. on Probab.
Vol.56 1988年
P1-120

of more general isotropic random walks on a homogeneous tree is stated. The proof uses the Plancherel-Godement theorem and the explicit form of the Plancherel measure on a tree. Finally, in Section 8, Sawyer's transience result is presented. Its statement on the rate of escape in the isotropic case goes far beyond the transience property. In the demonstration of this far-reaching assertion classical probabilistic arguments are combined with the techniques available in the theory of Gelfand pairs.

The principal reference of Sections 1 to 5 is Cartier's fundamental paper [4]. Our presentation is just a reorganization of the ideas of [4]. Concerning Martin boundaries of random walks on trees we emphasize the excellent contributions [13], [14] of M.A. Picardello and W. Woess in which geometrical descriptions of the Martin boundary of a tree are given. The theory of Laplacians on trees has been deepened by A. Koranyi and M.A. Picardello in [8]. Concerning the transience and recurrence properties of random walks on trees we only mention, for the purpose of orienting the reader, the expository paper [7] by P. Gerl and the references to his important work therein. In Sections 6 to 8 we shall refer also to the papers [1] and [6] by J.-P. Arnaud and J.-L. Dunau respectively. They appear as basic references to the profound studies of G. Letac ([10], [11]) within the framework of Gelfand pairs as developed in the survey articles [9] and [12]. The reader interested in a detailed exposition of the harmonic analysis of Gelfand pairs is recommended to consult Dieudonné's book [5]. The recent contribution [16] of S. Sawyer and T. Steger indicates what can be done if one

Sem. on Probab.
Vol.56 1988年
P1-120

wants to generalise the theory to anisotropic random walks.

I am grateful to my collaborators who have helped preparing and mending the material of this paper. Mr. R. Weck wrote a Master's thesis on the subject, and Dr. G. Turnwald offered a remarkable portion of constructive criticism. The difficult task of typing the manuscript was skillfully done by Mrs. E. Gugl. Special thanks I owe to Professor Y. Okabe who most graciously invited me to Sapporo and, in connection with my lecturing there, encouraged me to write up these notes. Professor M. Tsuchiya from the University of Kanazawa was kind enough to include this set of notes in the prestigious series of the "Probability Seminar".

Sem. on Probab.
Vol.56 1988年
P1-120

I The Martin compactification of a tree

§ 1 Geodesics in a tree

In this preparatory section we shall review some basic notions from graph theory with the aim of introducing geodesics in a tree.

1.1 Definition. A *graph* (S,A) consists of a set S and a set A of two-element subsets of S. The elements of S are called the *vertices* of (S,A), those of A its *edges*.

Two edges s,t of (S,A) are said to be *joined* if $\{s,t\} \in A$.

1.2 Definition. A graph (S,A) with $S \neq \emptyset$ is called a *tree* if the following conditions are satisfied:

(a) (S,A) is *connected* in the sense that for each two vertices $s,s' \in S$ there exists a sequence $\{s_o, s_1, \ldots, s_n\}$ of vertices in S such that $s_o = s$, $s_n = s'$ and $\{s_{i-1}, s_i\} \in A$ for all $i = 1,\ldots,n$.

(b) There does not exist any *nontrivial closed path* in S, i.e. a sequenc $\{s_1,\ldots,s_n\}$ of $n \geq 3$ pairwise different vertices in S such that $\{s_i, s_{i+1}\} \in A$ for all $i=1,\ldots,n-1$ and $\{s_n, s_1\} \in A$.

A tree (S,A) is said to be *infinite* if S is an infinite set, and *locally finite* if for every $s \in S$ there exist only finitely many vertices $t \in S$ such that $\{s,t\} \in A$.

1.3 General assumption for the entire chapter: (S,A) is an infinite, locally finite tree.

1.4 Definition. A sequence $c = [s_o,\ldots,s_n]$ with $s_o,\ldots,s_n \in S$ is called a *path* in (S,A) if $\{s_{i-1}, s_i\} \in A$ for all $i=1,\ldots,n$, and a *closed path* if in addition $s_o = s_n$.

For any path $c = [s_o,\ldots,s_n]$ in (S,A), $\alpha(c) := s_o$ is called the

Sem. on Probab.
Vol.56 1988年
P1-120

starting point, $\beta(c) := s_n$ the *end point* and $\ell(c) := n$ the *length* of c. The vertices $s_1,\dots,s_{n-1}$ are called *intermediate points*.

Paths in (S,A) of length 1 are named *arcs*.

1.5 Facts. If $c := [s_o,\dots,s_n]$ and $c' := [s'_o,\dots,s'_m]$ are paths in (S,A) and if $\alpha(c') = \beta(c)$, then the *product* of c and c' is the path

$$c\cdot c' := [s_o,\dots,s_{n-1},s_n,s'_1,\dots,s'_m]$$

with starting point $\alpha(c\cdot c') = s_o(=\alpha(c))$, end point $\beta(c\cdot c') = s'_m\ (=\beta(c'))$ and length $\ell(c\cdot c') = n+m$.

For every edge $\{s,t\} \in A$ there exist two arcs $[s,t]$ and $[t,s]$ obtained by *orientation* of $\{s,t\}$.

Any path $c = [s_o,\dots,s_n]$ admits a unique product representation

$$c = a_1\cdot\ldots\cdot a_n$$

with arcs $a_i := [s_{i-1},s_i]$ for $i=1,\dots,n$.

For every path $c = [s_o,\dots,s_n]$ the *opposite path* $[s_n,\dots,s_o]$ is abbreviated by c^-

Since (S,A) is connected, there are arbitrarily many paths with starting point s and end point t. We are going to study paths of smallest length in (S,A).

1.6 Theorem. Let $s,t \in S$. Then

(i) there exists exactly one path $c_o = [s_o,\dots,s_n]$ such that $s_o = s$, $s_n = t$ and $s_i \neq s_j$ for all $i,j = o,\dots,n$; $i\neq j$.

(ii) Given c_o and a path $c = [t_o,\dots,t_m]$ with $t_o=s$ and $t_m=t$, then there is a sequence $\{m_o,\dots,m_n\}$ in $\mathbf{Z}_+$ with $o = m_o < m_1 < \dots < m_n = m$ such that $t_{m_i} = s_i$ for all $i = 1,\dots,n$.

(iii) If, moreover, the vertices of c satisfy $t_{i-1} \neq t_{i+1}$ for all $i=1,\ldots,m-1$, then $c=c_o$.

Proof. 1. We shall show (ii) along with the statement of existence in (i).

Let $c = [t_o,\ldots,t_m]$ be an arbitrary path in (S,A) with $t_o = s$ and $t_m = t$. Then consider the set C of all paths c' of the form $[t_{m_o},\ldots,t_{m_r}]$ with $0 = m_o < m_1 < \ldots < m_r = m$. Since C is a finite set, we can choose a path $c_o = [t_{m_o},\ldots,t_{m_n}] \in C$ with $n := \min\{\ell(c'):c' \in C\}$. Put $s_i := t_{m_i}$ for all $i=1,\ldots,n$. It remains to be shown that $s_i \neq s_j$ for all $i,j=o,\ldots,n$, $i\neq j$. Suppose, on the contrary, that there exist i,j with $o \leqq i < j \leqq n$ such that $s_i = s_j$. Then $c_o' := [s_o,\ldots,s_i,s_{j+1},\ldots,s_n]$ is a path in C with $\ell(c_o')<n$ contradicting the definition of c_o.

2. The statement of uniqueness in (i) and (iii) will follow as soon as we have established the following assertion:

(iv) Let $c = [t_o,\ldots,t_m]$ be a path in (S,A) with $t_{i-1} \neq t_{i+1}$ for all $i=1,\ldots,m-1$ and let $c' = [t_o',\ldots,t_r']$ be a path with $t_i' \neq t_j'$ for $i,j = o,\ldots,r$, $i \neq j$ such that $t_o = t_o'$ and $t_m = t_r'$. Then $c = c'$.

Thus, under the assumption (iv), the uniqueness of (i) can be deduced as follows: Let c_o' be another path in (S,A) with pairwise different vertices, which joins s and t. Then c_o' satisfies in particular the property of c in (iv). Now apply (iv) to $c' := c_o$ with c_o of 1. Then (iv) implies that $c_o' = c_o$.

3. We shall show (iv) by induction on m.

For $m = o$ the assertion is clear, since $t_o = t_m$ together with the hypothesis yields $r = o$, i.e. $c = c'$.

Let m=1. If r>1, then $[t_o',\ldots,t_r',t_o]$ is a nontrivial closed path,

Sem. on Probab.
Vol.56 1988年
P1-120

a fact which contradicts the definition of a tree. Consequently $r=1$, and by $t_o = t'_o$, $t_1 = t'_1$ we obtain $c = c'$.

Let now $m \geqq 2$ and let (iv) be proved for all paths of length $<m$. If $t_{m-1} \neq t'_i$ for all $i=o,\ldots,r$ then by induction we get $[t_o,\ldots,t_{m-1}] = [t'_o,\ldots,t'_r, t_{m-1}]$ which implies $t_{m-2} = t'_r = t_m$, contradicting the hypothesis. Hence we obtain product representations $c=c_1 \cdot c_2$ and $c'=c'_1 \cdot c'_2$ with $t_{m-1} = \beta(c_1) = \beta(c'_1)$. Furthermore we have $\ell(c_1)<m$ and c_1, c'_1 satisfy the assumption in (iv). By the induction hypothesis it follows that $c_1=c'_1$. Similarly we find that $c_2=c'_2$ and hence $c=c'$. $\_\_|$

The preceding theorem justifies the following

1.7 <u>Definition:</u> Let $s,t \in S$. The path $c_o = [s_o,\ldots,s_n]$ with $s_{i-1} \neq s_{i+1}$ for all $i=1,\ldots,n-1$, which by the theorem exists uniquely, is called the *geodesic from* s *to* t; it will be abbreviated by $c(s,t)$.

1.8 <u>Notation.</u> For all $s,t \in S$ the integer

$$d(s,t) := \ell(c(s,t))$$

is called the *distance between* s *and* t.

1.9 <u>Properties.</u> Let $s,t \in S$.

1.9.1 $d(s,t) = 1$ iff $\{s,t\} \in A$.

1.9.2 $d(s,t) = 0$ iff $s = t$.

1.9.3 $d(s,t) = d(t,s)$.

An elementary discussion using Theorem 1.6 yields

1.9.4 that given $s, s', s'' \in S$ there exists a unique $t \in S$ which belongs to all geodesics $c(s,s')$, $c(s,s'')$ and $c(s',s'')$. For this vertex t we have

$c(s,s') = c(s,t).c(t,s')$,

$c(s,s'') = c(s,t).c(t,s'')$,

$c(s',s'') = c(s',t).c(t,s'')$,

and $c(t,s)$, $c(t,s')$ and $c(t,s'')$ admit only t as a common vertex.

The vertex t is called the *center* of s,s',s''.

1.9.5 $d(s,s'') \leqq d(s,s') + d(s',s'')$.

[From 1.9.4 we deduce the existence of a $t \in S$ such that

$$\begin{aligned} d(s,s'') &= \ell(c(s,s'')) \\ &= \ell(c(s,t)) + \ell(c(t,s'')) \\ &\leqq \ell(c(s,t)) + 2\ell(c(s',t)) + \ell(c(t,s'')) \\ &= \ell(c(s,s')) + \ell(c(s',s'')) \\ &= d(s,s') + d(s',s''). \end{aligned}$$]

1.10 <u>Résumé.</u> The mapping

$$(s,t) \to d(s,t) = \ell(c(s,t))$$

from $S \times S$ into $\mathbb{Z}_+$ is a metric on S.

Now we want to extend Definition 1.4 and Theorem 1.6 beyond finiteness.

1.11 <u>Definition.</u> A sequence $w = [s_o, s_1, \dots, s_n, \dots]$ with $s_n \in S$ for all $n \geqq o$ is called an *infinite path in* S if $d(s_i, s_{i+1}) = 1$ for all $i \geqq o$.

Clearly $\alpha(w) := s_o$ is called the *starting point* of w.

Sem. on Probab.
Vol.56 1988年
P1-120

An infinite path $w = [s_o, s_1, \ldots, s_n, \ldots]$ in S is said to be an *infinite chain* if $s_i \neq s_j$ holds for all i,j 0 with $i \neq j$.

1.12 An application of Theorem 1.6 yields for two given infinite chains $w = [s_o, \ldots, s_n, \ldots]$ and $w' = [s'_o, \ldots, s'_n, \ldots]$ in S the following three possibilities for the set $\mathfrak{M}$ of all common vertices of w and w'.

(1) $\mathfrak{M} = \emptyset$.

(2) $\mathfrak{M}$ is a *finite interval in* w *and* w' in the sense that there are $p,q,n \in \mathbb{Z}_+$ satisfying

$$s_{p+i} = s'_{q+i} \quad \text{for } i=o,\ldots,n$$

or $s_{p+i} = s'_{q+n-i}$ for $i=o,\ldots,n$

and

$$s_i \notin \mathfrak{M} \quad \text{for} \quad i=0,\ldots,p-1,\ i>p+n.$$

(3) $\mathfrak{M}$ is an *infinite interval in* w *and* w' in the sense that there exist $p,q \in \mathbb{Z}_+$ such that

$$s_{p+n} = s'_{q+n} \quad \text{for all} \quad n \geq o$$

and

$$s_o,\ldots,s_{p-1} \text{ (and hence } s'_o,\ldots,s'_{q-1}) \notin \mathfrak{M}.$$

This last possibility suggests the following

1.13 Definition. Two infinite chains w and w' in S are said to be *equivalent* (in symbols $w \sim w'$) if they admit infinitely many common vertices.

The corresponding equivalence classes are called *ends*. Its totality will be abbreviated by B.

1.14 Theorem. Let $s,s' \in S$ and let $w = [s_o, s_1, \ldots]$ be an infinite chain in S with $\alpha(w) = s$. Then there exists exactly one infinite chain w' in S with $\alpha(w') = s'$ and $w' \sim w$.

Sem. on Probab.
Vol.56 1988年
P1-120

<u>Proof.</u> Let $c(s,s') = [t_o, t_1, \ldots, t_m]$ and let

$$p := \max\{i : o \leqq i \leqq m \text{ and } t_i \text{ is a vertex of } w\}.$$

Then $t_p = s_q$ for at least one $q \geqq o$. The paths $[s_o, \ldots, s_q]$ and $[t_o, \ldots, t_p]$ join s and s_q, and we have $s_{i-1} \neq s_{i+1}$ for $i=1,\ldots,q-1$ by the definition of an infinite chain as well as $t_{i-1} \neq t_{i+1}$ for $i = 1,\ldots,p-1$ by the definition of a geodesic. By Theorem 1.6 we conclude that

$$[s_o, \ldots, s_q] = [t_o, \ldots, t_p],$$

in particular that $p = q$ and thus $s_j = t_j$ for all $j = o,\ldots,p$.

Therefore

$$w' = [t_m, t_{m-1}, \ldots, t_{p+1}, s_p, s_{p+1}, \ldots]$$

is an infinite chain in S with $\alpha(w') = s'$ such that $w' \sim w$. The uniqueness of w' follows from the fact that two equivalent infinite chains in S having the same starting point are identical by 1.12. ┘

We are now ready to extend Definition 1.7 to infinite chains.

1.15 <u>Definition.</u> Let $s \in S$ and $b \in B$. The infinite chain w in b with $\alpha(w) = s$, which by the theorem exists uniquely, is called the *geodesic from* s *to* b; it will be denoted by $w(s,b)$.

Sem. on Probab.
Vol.56 1988年
P1-120

§ 2 Green kernel and potentials

Now we are entering the potential theory of a tree. Simple properties of the Green kernel will yield an analogue of the Riesz decomposition theorem.

2.1 Notation. As in the preceding section we are given an infinite, locally finite tree (S,A) which from now on will be abbreviated just by S.

Let

Γ := the set of all paths in S (of finite length) and

$\Gamma(n) := \{c\in\Gamma : \ell(c) = n\}$ for all $n\in\mathbb{Z}_+$;

in particular we have that

$\Gamma(1)$ = the set of all arcs.

Moreover, let

$\Gamma_{s,t} := \{c\in\Gamma : \alpha(c) = s, \beta(c) = t\}$ and

$\Gamma_{s,t}(n) := \Gamma(n)\cap\Gamma_{s,t}$ for all $s,t\in S$, $n\in\mathbb{Z}_+$.

2.2 General Assumption.

2.2.1 There exists a mapping $p : \Gamma(1)\to\mathbb{R}_+^\times$. Putting

$$p(c) := \prod_{i=1}^{n} p(a_i)$$

for any $c = [s_o,\ldots,s_n]\in\Gamma(n)$ with $a_i := [s_{i-1},s_i]\in\Gamma(1)$ $(i=1,\ldots,n)$

Sem. on Probab.
Vol.56 1988年
P1-120

p extends to $\Gamma(n)$ for every $n \geq 1$ and thus to the whole of Γ.

In particular one obtains

$$p(c) = 1 \quad \text{for all } c \in \Gamma(o),$$

and if the product $c.c'$ is defined for $c \in \Gamma(n)$, $c' \in \Gamma(m)$ with $n,m \geq o$, then

$$p(c.c') = p(c).p(c').$$

2.2.2 For all $s,t \in S$

$$\sum_{c \in \Gamma_{s,t}} p(c) < \infty$$

holds.

2.3 Definition. The *mean-value* and *Laplacian operators* are defined on the space $\mathfrak{F}(S,\mathbb{R})$ of real-valued functions h on S by

$$Ph := \sum_{\{t \in S : d(\cdot,t)=1\}} p([\cdot,t])\, h(t)$$

and

$$\Delta h := (P - I)\, h := Ph - Ih$$

respectively.

Here I denotes the identity operator on $\mathfrak{F}(S,\mathbb{R})$.

2.4 Definition. A function $h \in \mathfrak{F}(S,\mathbb{R})$ is said to be *superharmonic* if

$$\Delta h \leq o,$$

and *harmonic* if

Sem. on Probab.
Vol.56 1988年
P1-120

$\Delta h = 0$.

In order to extend the notion of the mean-value operator to more general functions p we need some

2.5 Preparations about kernels.

A *kernel* on S will be any mapping $U : S\times S \to \overline{\mathbb{R}}_+$.

For two kernels U and V on S the *product* $U\cdot V$ is defined by

$$U\cdot V(s,s'') := \sum_{s'\in S} U(s,s')\cdot V(s',s'')$$

whenever $s,s''\in S$. Clearly multiplication of kernels on S is associative.

There is the *unit kernel* I on S defined by

$$I(s,t) := \delta_{s,t}$$

for all $s,t\in S$; it takes the part of the *neutral element* in the multiplicative semigroup of kernels on S.

2.6 Kernels attached to subsets of Γ.

For any subset C of Γ and $s,t\in S$ we introduce the set

$$C_{s,t} := \{c\in C : \alpha(c) = s,\ \beta(c) = t\},$$

and we define the kernel U_C on S by

$$U_C(s,t) := \sum_{c\in C_{s,t}} p(c)$$

for all $s,t\in S$. Since General Assumption 2.2 implies that

$$U_C(s,t) \leqq \sum_{c\in\Gamma_{s,t}} p(c) < \infty$$

for all $s,t\in S$, U_C is in fact a finite kernel on S.

Let C,C' and C" be subsets of Γ. We assume that every path of the form $c'\cdot c''$ with $c'\in C'$ and $c''\in C''$ belongs to C and conversely that

Sem. on Probab.
Vol.56 1988年
P1-120

every $c \in C$ admits a unique product representation of the form $c = c'.c''$ with $c' \in C'$ and $c'' \in C''$.

Then

$$U_C = U_{C'} \cdot U_{C''} .$$

[Let $s,u \in S$. For each $t \in S$ we consider the subset

$$D_t := \{c \in \Gamma : c = c' \cdot c'' \text{ with } c' \in C'_{s,t}, c'' \in C''_{t,u}\}$$

of C and the mapping $\phi_t : C'_{s,t} \times C''_{t,u} \to D_t$ defined by

$$\phi_t(c',c'') := c' \cdot c''$$

for all $c' \in C'_{s,t}, c'' \in C''_{t,u}$. By assumption ϕ_t is bijective for all $t \in S$, and for each $u \in S$,

$$C_{s,u} = \bigcup_{t \in S} D_t$$

with pairwise disjoint sets D_t $(t \in S)$. Then for all $(s,u) \in S \times S$,

$$\begin{aligned} U_C(s,u) &= \sum_{c \in C_{s,u}} p(c) \\ &= \sum_{t \in S} \sum_{c \in D_t} p(c) \\ &= \sum_{t \in S} \sum_{c' \in C'_{s,t}} \sum_{c'' \in C''_{t,u}} p(c'.c'') \\ &= \sum_{t \in S} \sum_{c' \in C'_{s,t}} \sum_{c'' \in C''_{t,u}} p(c') \cdot p(c'') \\ &= \sum_{t \in S} U_{C'}(s,t) U_{C''}(t,u) \end{aligned}$$

which implies the assertion.]

2.7 Distinguished kernels on S.

2.7.1 Let $C := \Gamma(o)$. Then

$$N^o := U_{\Gamma(o)}$$

Sem. on Probab.
Vol.56 1988年
P1-120

is the unit kernel on S.

2.7.2 Let $C := \Gamma(1)$. Then

$$N := N^1 := U_{\Gamma(1)}$$

denotes the kernel

$$(s,t) \to N(s,t) = \begin{cases} p([s,t]) & \text{if } d(s,t) = 1 \\ 0 & \text{otherwise .} \end{cases}$$

2.7.3 Let $C := \Gamma(2)$. Then

$$N^2 := N \cdot N = U_{\Gamma(2)}$$

is the kernel

$$(s,t) \to N^2(s,t) = \sum_{c \in \Gamma_{s,t}(2)} p(c)$$

$$= \begin{cases} \sum_{\{u \in S: d(s,u)=d(u,t)=1\}} p([s,u]) \cdot p([u,t]) & \text{if } d(s,t) \in \{o,2\} \\ 0 & \text{otherwise} \end{cases}$$

[Every path of length 2 can be uniquely written as the product of two arcs.]

More generally, putting

$$N^n := U_{\Gamma(n)}$$

for all $n>2$, an induction argument shows that

$$N^n = \prod_{i=1}^{n} N_i$$

with $N_i := N$ for all $i=1,\ldots,n$.

2.7.4 Let $C := \Gamma$. Then

$$U_\Gamma(s,t) = \sum_{c \in \Gamma_{s,t}} p(c)$$

for all $s,t \in S$. We define

Sem. on Probab.
Vol.56 1988年
P1-120

$$G := U_\Gamma$$

to be the *Green kernel* on S *corresponding to* p.

Since $\Gamma = \bigcup_{n\geq o} \Gamma(n)$ is a pairwise disjoint union and p is positive, one obtains

$$G = \sum_{n\geq o} N^n .$$

By General Assumption 2.2 and the connectivity of the tree S we conclude that G is positive and finite. Moreover one has the equalities

$$G = I + N\cdot G = I + G\cdot N.$$

2.7.5 Let $C := K := \{c\in\Gamma : \alpha(c) = \beta(c)\}$. The kernel

$$L := U_K$$

will be of special importance.

Similarly the kernel

2.7.6

$$F := U_\Lambda$$

for $\Lambda := \{c = [s_o,\dots,s_n] : n>o,\ s_i \neq s_n \text{ for all } i=1,\dots,n-1\}$.

Since S is connected, the sets $\Lambda_{s,t}$ are nonempty for all $s,t \in S$, and hence F is positive.

Now we shall study properties of the kernels G and F.

2.8 Proposition. For all $s,t \in S$,

$$G(s,t) = I(s,t) + F(s,t)\cdot G(t,t).$$

Proof: Every path $c = [s_o,\dots,s_n]$ of length $n>o$ can be written as a product $c_1\cdot c_2$ with $c_1 \in \Lambda$ and $c_2 \in K$; one just puts

Sem. on Probab.
Vol.56 1988年
P1-120

$$\beta(c_1) = \alpha(c_2) = s_k$$

with

$$k := \min\{\ell : 1 \leq \ell \leq n,\ s_\ell = s_n\}.$$

Then the discussion in 2.6 yields that

$$G = U_\Gamma = U_{\Gamma(o)} + U_{\Gamma \setminus \Gamma(o)}$$

$$= I + F \cdot L,$$

and since

$$L(s,t) = \sum_{c \in K_{s,t}} p(c) = \delta_{s,t} \cdot G(s,s),$$

we obtain

$$F \cdot L(s,t) = \sum_{s' \in S} F(s,s')L(s',t)$$

$$= F(s,t) \cdot G(t,t)$$

whenever $s,t \in S$. ┘

2.9 Proposition. For pairwise different vertices s,s' and t of S such that $t \in c(s,s')$ one has

$$F(s,s') = F(s,t) \cdot F(t,s').$$

Proof. For all $c \in \Lambda_{s,t}$ and $c' \in \Lambda_{t,s'}$ we set

$$\phi(c,c') := c \cdot c'.$$

If we have shown that ϕ is a bijection from $\Lambda_{s,t} \times \Lambda_{t,s'}$ onto $\Lambda_{s,s'}$, the chain of equalities

$$F(s,s') = \sum_{c'' \in \Lambda_{s,s'}} p(c'')$$

$$= \sum_{c \in \Lambda_{s,t}} \sum_{c' \in \Lambda_{t,s'}} p(c \cdot c')$$

$$= \sum_{c \in \Lambda_{s,t}} \sum_{c' \in \Lambda_{t,s'}} p(c)p(c')$$

Sem. on Probab.
Vol.56 1988年
P1-120

$$= F(s,t).F(t,s')$$

yields the assertion.

1. Let $c \in \Lambda_{s,t}$, $c' \in \Lambda_{t,s'}$. If $s' \notin c$ then by definition $\phi(c,c') = c.c' \in \Lambda_{s,s'}$. For $s' \in c$ we have $c = c_1 . c_2$ with $c_1 \in \Gamma_{s,s'}$ and $c_2 \in \Gamma_{s',t}$. By Theorem 1.6 (ii) every path joining s and s' contains the vertices of c(s,s'). In particular, c_1 contains t. Hence $c \in \Lambda_{s,t}$ implies $c_2 = [t]$ and $t = s'$ contradicting the hypothesis $t \neq s'$.

2. Now let $c := [s_o, \ldots, s_n] \in \Lambda_{s,s'}$. Again by Theorem 1.6 (ii), $c \ni t$. Consider $c = c_1 \cdot c_2$ such that $\beta(c_1) = \alpha(c_2) = s_k$ with $k := \min\{\ell : o < \ell < n, s_\ell = t\}$. Then $c = c_1 \cdot c_2$ with $c_1 \in \Lambda_{s,t}$ by the very choice of $\beta(c_1)$ and $c_2 \in \Lambda_{t,s'}$ since $c \in \Lambda_{s,s'}$. The product representation is unique.]

<u>2.10 Theorem</u>. For all $s,s' \in S$ and a vertex t of c(s,s'),

$$G(s,s') = G(s,t) \cdot G(t,t)^{-1} \cdot G(t,s').$$

<u>Proof.</u> For t=s or t=s' or s=s' the formula is obviously true. Let therefore s,s' and t be pairwise different vertices. From Proposition 2.8 we infer the formulae

(a) $G(s,s') = F(s,s') \cdot G(s',s')$,

(b) $G(s,t) = F(s,t) \cdot G(t,t)$

and

(c) $G(t,s') = F(t,s') \cdot G(s',s')$.

Proposition 2.9 provides us with the formula

(d) $F(s,s') = F(s,t) \cdot F(t,s')$

Sem. on Probab.
Vol.56 1988年
P1-120

which yields the assertion. _|

Let U be a kernel on S. U can be viewed as an operator on the space $\mathfrak{F}(S,\overline{\mathbb{R}}_+)$ of numerical functions h on S via

$$Uh := \sum_{t\in S} U(\cdot,t)\cdot h(t) .$$

Clearly

$$Ih = h$$

and

$$(U\cdot V)h = U(Vh)$$

for all $h \in \mathfrak{F}(S,\overline{\mathbb{R}}_+)$. In particular,

$$Nh = \sum_{\{t\in S:d(\cdot,t)=1\}} p([\cdot,t])h(t)$$

$$= Ph.$$

2.11 Definition. For every $v \in \mathfrak{F}(S,\overline{\mathbb{R}}_+)$ the function Gv is called the *potential* of v.

2.12 Theorem (Riesz decomposition). For any function $g \in \mathfrak{F}(S,\mathbb{R}_+)$ the following statements are equivalent:

(i) g is superharmonic.

(ii) There exist a function $v \in \mathfrak{F}(S,\mathbb{R}_+)$ and a harmonic function $h \in \mathfrak{F}(S,\mathbb{R}_+)$ such that

$$g = Gv + h$$

holds.

The decomposition in (ii) is unique.

Proof. 1. (i) => (ii). From $G = I+N.G$ we conclude that $\Delta G = N.G-G=-I$. Hence the representation $g = Gv+h$ implies $-\Delta g=v$ if h is harmonic,

Sem. on Probab.
Vol.56 1988年
P1-120

i.e. such that $\Delta h = o$. This shows already the uniquenes part of (ii). To prove the existence we note that $v := -\Delta g$ is $\geqq o$ if g is superharmonic.

Setting $h := g-Gv$ we arrive at

$$\Delta h = \Delta g+\Delta.Gv = \Delta g+v = o.$$

From $g = v+Ng$ we obtain by induction that

$$g = \sum_{i=o}^{n} N^i v+N^{n+1} g.$$

Consequently

$$g \geqq \sum_{i=o}^{n} N^i v$$

for all $n \geqq o$. Thus

$$g \geqq \sum_{i\geqq o} N^i v = Gv,$$

i.e. $h \geqq o$.

2. (ii) => (i). If $g = Gv+h$ is given with $v \geqq o$ and harmonic h then g is in fact superharmonic, since $\Delta g = \Delta.Gv = -v \leqq o$. _|

Sem. on Probab.
Vol.56 1988年
P1-120

§ 3 The natural compactification

In classical potential theory one studies the boundary $\partial\mathbb{D}$ of the open unit disc $\mathbb{D}$ in $\mathbb{C}$ as the natural compactification of $\mathbb{D}$. To every point z of $\partial\mathbb{D}$ there is attached a radius starting from the centre 0 of $\mathbb{D}$ which is the geodesic between o and z. We are going to develop a similar idea for a tree by using geodesics starting from a fixed vertex. The space of all such geodesics will serve as a boundary of the tree.

Given an infinite, locally finite tree S we want to use the space Σ_o of all infinite chains starting at o in order to construct a compactification of S.

For every $n \geq o$ let

$$S_n := \{s \in S : d(o,s) = n\}.$$

In particular we get

$$S_o = \{o\}.$$

3.1 Observation. For every geodesic

$$c(o,s) := [s_o,\dots,s_n]$$

with $s \in S_n$ for $n \geq o$ there exists exactly one vertex $s' \in S_{n-1}$ such that $d(s',s) = 1$.

[Clearly $s_k \in S_k$ for all $k=o,\dots,n$ by Theorem 1.6 and the properties of the metric d. In particular $s_{n-1} \in S_{n-1}$ and $d(s_{n-1},s) = 1$. Now let $t \in S$ with $t \neq s_{n-1}$ and $d(t,s) = 1$. Again by Theorem 1.6, the path $c := [s_o,\dots,s_n,t]$ equals $c(o,t)$, since $s_{i-1} \neq s_{i+1}$ for $i=1,\dots,n-1$ and $t \neq s_{n-1}$ by assumption. Thus $t \in S_{n+1}$, and the choice of s' is unique.

We write $\pi_n(s)$ instead of s' and note that in any representation of $c(s,o)$ as a product of n arcs, $[s,\pi_n(s)]$ is the first one.

Sem. on Probab.
Vol.56 1988年
P1-120

3.2 Definition of a *projective system.*

Consider the chain

$$S_o \overset{\pi_1}{\longleftarrow} S_1 \overset{\pi_2}{\longleftarrow} S_2 \overset{\pi_3}{\longleftarrow} \dots \overset{\pi_{n-1}}{\longleftarrow} S_{n-1} \overset{\pi_n}{\longleftarrow} S_n \overset{\pi_{n+1}}{\longleftarrow} \dots \quad .$$

For every $n \geqq 1$ and any choice of $s \in S_{n-1}$ the set $\pi_n^{-1}(s)$ is finite, since the tree S is assumed to be locally finite. Consequently all sets S_n are finite ($n \geqq o$). Moreover, all sets S_n are nonempty ($n \geqq o$), since the disjoint union $S = \bigcup_{n \geqq o} S_n$ is an infinite set by hypothesis and $S_n = \emptyset$ for some $n \geqq 1$ would imply, by definition of π_{n+1}, that $S_{n+1} = \emptyset$.

3.3 Definition of the *projective limit*

$$\lim_{\longleftarrow} S_n :$$
$$= \{w = [s_o, \dots, s_n, \dots] : s_n \in S_n \text{ for all } n \geqq o,$$
$$\pi_n(s_n) = s_{n-1} \text{ for all } n \geqq 1\} .$$

Obviously

$$\lim_{\longleftarrow} S_n = \Sigma_o \quad .$$

In order to render the theory independent of the starting point o we relate the space Σ_o to the space B of ends of S.

3.4 Proposition. The mapping $\mathfrak{I} : B \to \Sigma_o$ defined by

$$\mathfrak{I}(b) := w(o,b)$$

for all $b \in B$ is a bijection.

Proof. From the definition of the geodesic $w(o,b)$ as an infinite chain with starting point o we infer that $w(o,b) \in \Sigma_o$. Let $b_1, b_2 \in B$ and $\mathfrak{I}(b_1) = \mathfrak{I}(b_2)$, i.e. $b_1 \ni w(o,b_1)$, $b_2 \ni w(o,b_2)$ and $w(o,b_1) = w(o,b_2)$. Then $b_1 = b_2$, since two equivalence classes are either equal or disjoint.

Thus $\mathfrak{I}$ is injective.

Let now

$$w = [s_o, \dots, s_n, \dots] \in \Sigma_o .$$

Then there exists a $b \in B$ with $w \in b$. From Theorem 1.14 we conclude $w = w(o,b)$, which implies that $\mathfrak{I}(b) = w$, and hence that $\mathfrak{I}$ is surjective. _|

3.5 <u>Extension of the bijection</u> $\mathfrak{I}$ from B to $\hat{S} := S \cup B$.

Let $o \in S$ remain fixed. We define for every $n \geq o$ the set

$$S(n) := \bigcup_{i=o}^{n} S_i .$$

Then

$$S = \bigcup_{n \geq o} S(n)$$

such that $S(n)$ is finite $\neq \emptyset$ for every $n \geq o$.

For all $n \geq 1$ we introduce the mappings $\pi(n) : S(n) \to S(n-1)$ by

$$\pi(n)(s) := \begin{cases} \pi_n(s) & \text{if } s \in S_n \\ s & \text{if } s \in S(n-1). \end{cases}$$

In this way we obtain a projective system

$$S(o) \overset{\pi(1)}{\longleftarrow} S(1) \overset{\pi(2)}{\longleftarrow} S(2) \longleftarrow \dots \overset{\pi(n-1)}{\longleftarrow} S(n-1) \overset{\pi(n)}{\longleftarrow} S(n) \longleftarrow \dots$$

with projective limit

$$\begin{aligned} \Sigma_o^* &:= \varprojlim S(n) \\ &= \{(s_o, \dots, s_n, \dots) : s_n \in S(n) \text{ for all } n \geq o, \\ &\qquad \pi(n)(s_n) = s_{n-1} \text{ for all } n \geq 1\} . \end{aligned}$$

The mapping $\mathfrak{I}^* : \hat{S} \to \Sigma_o^*$ will be defined by

$$\mathfrak{I}^*(x) := \begin{cases} (s_o, s_1, \dots, s_n, s_n, s_n, \dots) & \text{if } x \in S,\ c(o,x) = [s_o, \dots, s_n] \\ (s_o, s_1, \dots, s_n, s_{n+1}, \dots) & \text{if } x \in B,\ w(o,x) = [s_o, \dots, s_n, \dots] \end{cases}$$

In the sequel we shall extend the symbol $w(o,x)$ to *geodesics from* $o \in S$ *to any* $x \in \hat{S}$.

3.6 Proposition. The mapping $\mathfrak{I}^*$ is bijective, and

$$\mathrm{Res}_B \mathfrak{I}^* = \mathfrak{I}.$$

Proof. The fact that $\mathrm{Res}_B \mathfrak{I}^* = \mathfrak{I}$ is clear by definition; the bijectivity of $\mathrm{Res}_B \mathfrak{I}^*$ follows from Poposition 3.4. It remains to be shown that also

$$\mathfrak{I}^*|_S : S \to \Sigma_o^* \setminus \Sigma_o$$

is bijective. Let $s,t \in S$ such that $\mathfrak{I}^*(s) = \mathfrak{I}^*(t)$. Then $c(o,s) = c(o,t)$ i.e. $s = t$, and this shows the injectivity of $\mathfrak{I}^*|_S$. Now let

$$\sigma := (s_o, \ldots, s_n, \ldots) \in \Sigma_o^* \setminus \Sigma_o .$$

Then we obtain for all $j \geqq 1$ that

$$\pi(j)(s_j) = \begin{cases} s_{j-1} \in S_{j-1} & \text{if } s_j \in S_j \\ s_j & \text{if } s_j \in S(j-1) . \end{cases}$$

Since $\sigma \in \Sigma_o^* \setminus \Sigma_o$, there exists an $n \in \mathbf{Z}_+$ with $s_j \in S_j$ for all $j = o, \ldots, n$ and $s_k = s_n$ for all $k > n$. Hence $\sigma = \mathfrak{I}^*(s_n)$, and the surjectivity of $\mathfrak{I}^*|_S$ has been proved. ⌋

3.7 Orientation of an arc

3.7.1 for a fixed vertex.

Let s be a fixed vertex of a tree S. We define

$$\delta_s(t,t') := d(t,s) - d(t',s)$$

for all $t, t' \in S$. Then for $t,t' \in S$ satisfying $d(t,t') = 1$ we get

$$\delta_s(t,t') \in \{-1,1\} .$$

[Clearly $\delta_s(t,t') = -\delta_s(t',t)$. Without loss of generality we suppose

Sem. on Probab.
Vol.56 1988年
P1-120

that $t \neq s$ and $t' \neq s$. But then

$$|\delta_s(t,t')| = |d(t,s) - d(t',s)| \leqq d(t,t') = 1,$$

whence $\delta_s(t,t') \in \{-1,0,1\}$. Now we assume that $\delta_s(t,t') = o$, i.e. $d(t,s) = d(t',s)$. Let

$$c(s,t) = [s_o,\ldots,s_n]$$

for some $n \geqq 1$. Since $d(t',t) = 1$, we have either

$$[s_o,\ldots,s_{n-1}] = c(s,t') \quad (\text{in the case of } t' = s_{n-1})$$

or

$$[s_o,\ldots,s_n,t'] = c(s,t') \quad (\text{if } t' \neq s_{n-1});$$

this is implied by Theorem 1.6 (iii). But in both cases the fact that

$$n = d(s,t) \neq d(s,t') \in \{n-1,n+1\},$$

yields a contradiction. Thus $\delta_s(t,t') = o$ cannot be true, and the assertion has been established.]

Notation. If $t,t' \in S$ with $d(t,t') = 1$ and if

$$\delta_s(t,t') = 1 \quad ,$$

we will say that *the arc* $[t,t']$ *points in the direction of* (or towards) s.

Since $\delta_s(t,t') = -\delta_s(t',t)$, exactly one of the arcs $[t,t']$ and $[t',t]$ points towards s.

The set of all arcs in S which point toward s will be called an *orientation on the* tree S, more precisely the *orientation towards the centre* s.

3.7.2 for a fixed end.

Let b be a fixed end, and let

Sem. on Probab.
Vol.56 1988年
P1-120

$$w(s,b) = [s_o, \ldots, s_n, \ldots]$$

and

$$w(s',b) = [s'_o, \ldots, s'_n, \ldots]$$

the geodesic from s and s' respectively to b. Then there exist numbers $p,q \in \mathbb{N}$ with

$$s_{p+j} = s'_{q+j}$$

for all $j \geq o$ such that the vertices $s_o, \ldots, s_p, s'_o, \ldots, s'_{q-1}$ are pair-wise different; see Application 1.12. Now we define

$$\delta_b(s,s') := p - q$$

for all $s,s' \in S$. Then for $s,s' \in S$ satisfying $d(s,s') = 1$ we get

$$\delta_b(s,s') \in \{-1,1\}.$$

[Evidently $\delta_b(s,s') = -\delta_b(s',s)$. The inequality

$$|\delta_b(s,s')| = |p-q| \leq p+q = d(s,s') = 1$$

implies that $\delta_b(s,s') \in \{-1,o,1\}$. Since $\delta_b(s,s') = p-q \equiv p+q\,(2)$, we have $\delta_b(s,s') = \pm 1$.]

<u>Notation.</u> If $s,s' \in S$ with $d(s,s') = 1$ and if

$$\delta_b(s,s') = 1 ,$$

then $p=1$ and $q=o$, whence

$$w(s',b) = [s_1, \ldots, s_n, \ldots]$$

if $\quad w(s,b) = [s_o, \ldots, s_n, \ldots]$.

Sem. on Probab.
Vol.56 1988年
P1-120

In this case we will say that *the arc* $[s,s']$ *points in the direction of* (or towards) b .

Since $\delta_b(s,s') = -\delta_b(s',s)$, exactly one of the arcs $[s,s']$ and $[s',s]$ points towards b.

The set of all arcs in S which point towards b defines the *orientation towards the centre* b.

Let $\hat{\mathfrak{G}}$ denote the topology on $\hat{S}$ generated by the system

$$\{U(s,t) : s,t \in S \text{ with } d(s,t) = 1\}$$

of sets

$$U(s,t) := \{x \in \hat{S} : \delta_x(s,t) = 1\}.$$

3.8 Proposition. For each $n \geq o$ and every $s \in S_n$ let

$$V_n(s) := \{x \in \hat{S} : w(o,x) \ni s\}.$$

The system consisting of these sets $V_n(s)$ and their complements generates the topology $\hat{\mathfrak{G}}$ on $\hat{S}$.

Proof. (i) Since for any $s,t \in S$ with $d(s,t) = 1$ we have either $\delta_o(s,t) = 1$ or $= -1$ and since

$$\begin{aligned} U(t,s) &= \{x\in\hat{S} : \delta_x(t,s) = 1\} \\ &= \{x\in\hat{S} : \delta_x(s,t) = -1\} \\ &= \hat{S} \setminus \{x\in\hat{S} : \delta_x(s,t) = 1\} \\ &= \hat{S} \setminus U(s,t) , \end{aligned}$$

the topology $\hat{\mathfrak{G}}$ is generated by the system of sets $U(s,t)$ and $U(t,s) = \hat{S} \setminus U(s,t)$ where $[s,t]$ runs through the set of arcs with $\delta_o(s,t)=1$.

(ii) The sets $V_n(s)$ with $s \in S_n$, $n \geq o$, enjoy the following properties:

Sem. on Probab.
Vol.56 1988年
P1-120

(a) $V_o(o) = \hat{S}$

(b) $V_n(s) = \{x \in \hat{S} : \delta_x(\pi_n(s), s) = 1\}$

$= U(\pi_n(s), s)$

(c) $\hat{S} \setminus V_n(s) = \hat{S} \setminus U(\pi_n(s), s)$

$= U(s, \pi_n(s))$,

where the latter equality has been established in (i).

(iii) Let conversely $s, s' \in S$ be given with $\delta_o(s, s') = 1$. Then there exists an $n \geq 1$ satisfying $s \in S_n$ and $s' = \pi_n(s)$, and

$$U(s, s') = \hat{S} \setminus V_n(s),$$

whence

$$\hat{S} \setminus U(s, s') = V_n(s). \;\lrcorner$$

In order to obtain useful properties of the topological space $(\hat{S}, \hat{\mathcal{O}})$ we first study the topological space $(\Sigma_o^*, \mathcal{O}^*)$, where the topology $\mathcal{O}^*$ is induced from the product topology of the space $\prod_{n \geq o} S(n)$ (the finite sets $S(n)$ carrying the discrete topology). By well-known facts from general topology $(\Sigma_o^*, \mathcal{O}^*)$ is easily seen to be a compact, totally disconnected, metrizable space.

3.9 <u>Proposition.</u> The bijection $\mathfrak{I}^*$ is a homeomorphism between the topological spaces $(\hat{S}, \hat{\mathcal{O}})$ and $(\Sigma_o^*, \mathcal{O}^*)$.

<u>Proof.</u> By Proposition 3.6, $\mathfrak{I}^*$ is bijective. It remains to be proved that the mappings $\mathfrak{I}^*$ and $\phi := \mathfrak{I}^{*-1}$ are continuous.

1. We show that ϕ is continuous. Let $n \geq o$, $s \in S_n$ and $c(o,s) = [s_o, \ldots, s_{n-1}]$. Then

$\mathfrak{I}^*(V_n(s))$

$= \mathfrak{I}^*(\{x \in \hat{S} : w(o,x) \ni s\})$

Sem. on Probab.
Vol.56 1988年
P1-120

$$= \{(t_o,\dots,t_n,\dots) \in \Sigma_o^* : t_n = s\}.$$

The set $M := \mathfrak{I}^*(V_n(s))$ is open and closed in Σ_o^*, since

$$M = \Sigma_o^* \cap M'$$

where

$$M' := (\prod_{m=o}^{n-1} S(m)) \times \{s\} \times (\prod_{m>n} S(m))$$

is open and closed in $\prod_{n\geq o} S(n)$. Thus for $n \geq o$ and $s \in S_n$ we have that

$$\phi^{-1}(V_n(s)) = \mathfrak{I}^*(V_n(s)) = M$$

is open in Σ_o^* and that

$$\phi^{-1}(\hat{S} \setminus V_n(s)) = \mathfrak{I}^*(\hat{S} \setminus V_n(s)) = \Sigma_o^* \setminus M$$

is open in Σ_o^*, since M is also closed in Σ_o^*. By Proposition 3.8 $\phi = \mathfrak{I}^{*-1}$ is therefore continuous.

2. We show that $\hat{S}$ is a Hausdorff space.

2a. Let $s, s' \in S$, $s \neq s'$ and let $c(s,s') = [s_o,\dots,s_n]$. Then $n \geq 1$, $s \in U(s_1,s_o)$ and $s' \in U(s_o,s_1) = \hat{S} \setminus U(s_1,s_o)$.

2b. Let $b, b' \in B$, $b \neq b'$. Then there exist $s_1, s_2 \in S_1$ with

$$\delta_b(o,s_1) = \delta_{b'}(o,s_2) = 1 .$$

If $s_1 \neq s_2$, then $b \in V_1(s_1)$, $b' \in V_1(s_2)$, and $V_1(s_1) \cap V_1(s_2) = \emptyset$.

Let $s_1 = s_2$. Since $w(o,b)$ and $w(o,b')$ admit only finitely many common vertices, there exists by 1.13 an $s \in S_n$ for $n > 1$ such that $w(o,b) \ni s$ and $w(o,b') \not\ni s$. But then $b \in V_n(s)$ and $b' \in \hat{S} \setminus V_n(s)$.

2c. Let $s \in S_n$ for $n \geq o$ and $b \in B$.

Let $b \in U(s,\pi_n(s))$. Then $s \in U(\pi_n(s),s) = \hat{S} \setminus U(s,\pi_n(s))$.

Now let $b \in U(\pi_n(s),s)$. Put

Sem. on Probab.
Vol.56 1988年
P1-120

$$w(\pi_n(s),b) = [s_o,s_1,s_2,\ldots].$$

Then $s_o = \pi_n(s)$, $s_1 = s$. Hence $s \in U(s_2,s)$ and $b \in U(s,s_2) = \hat{S} \setminus U(s_2,s)$. The assertion follows.

3. We show the continuity of $\mathfrak{S}^*$. By 1. ϕ is continuous, by 2. $\hat{S}$ is Hausdorff, and we have seen that Σ_o^* is compact. Moreover by Proposition 3.6 ϕ is bijective. We conclude that ϕ is a homeomorphism and so $\mathfrak{S}^* = \phi^{-1}$ is continuous. _|

3.10 <u>Theorem.</u>

(i) $(\hat{S},\hat{\mathfrak{O}})$ is a compact, totally disconnected, metrizable space which contains $(S,\mathfrak{P}(S))$ as an open dense subspace.

(ii) The mappings $\mathfrak{S}$ and $\mathfrak{S}^*$ are homeomorphisms from the closed subspace B and $\hat{S}$ onto the projective limits Σ_o and Σ_o^* respectively.

From (i) follows that $(\hat{S},\hat{\mathfrak{O}})$ is a *compactification* of $(S,\mathfrak{P}(S))$.

<u>Proof.</u> We have already shown that $(\hat{S},\hat{\mathfrak{O}})$ is a compact, totally disconnected, metrizable space.

1. We show that S is open in $\hat{S}$. Let $s \in S$ be fixed. For every $t \in S$ with $d(s,t) = 1$ we have $\delta_s(t,s) = 1$, i.e. $s \in U(t,s)$. Let $s' \in S$, $s' \neq s$ and put

$$c(s,s') := [s_o,\ldots,s_n].$$

Clearly $s' \notin U(s_1,s)$. For an arbitrary $b \in B$ with

$$w(s,b) = [s_o',\ldots,s_n',\ldots]$$

we have $b \notin U(s_1',s)$. Thus

$$\bigcap \{U(t,s) : t \in S \text{ with } d(t,s) = 1\} = \{s\}.$$

Since the set $\{t \in S : d(s,t) = 1\}$ is finite and $U(t,s)$ is open in $\hat{S}$ (by definition of $\hat{\mathfrak{O}}$), $\{s\}$ is open in $\hat{S}$, and hence also $S = \bigcup_{s \in S} \{s\}$ is open in $\hat{S}$. In particular $\hat{\mathfrak{O}}$ induces on S the discrete topology, and

Sem. on Probab.
Vol.56 1988年
P1-120

$B = \hat{S} \setminus S$ is closed in $\hat{S}$.

2. The following fact will be applied in the course of the proof: Let $b \in B$ and let

$$w(o,b) = [s_o, \ldots, s_n, \ldots].$$

The sequence $(V_n(s_n))_{n \geq o}$ forms a basis of open neighborhoods of b with

$$V_{n+1}(s_{n+1}) \subset V_n(s_n)$$

for all $n \geq o$.

In fact,

$$\mathfrak{F}^*(b) = (s_o, \ldots, s_n, \ldots)$$

and

$$V_n(s_n) = \{x \in \hat{S} : s_n \text{ is the n-th component of } \mathfrak{F}^*(x)\}.$$

This already implies the assertion if one utilizes Proposition 3.9; one just has to realize that the sequence of sets

$$\{(t_o, \ldots, t_n, \ldots) \in \Sigma_o^* : t_n = s_n\} \quad (n \geq o)$$

is a basis of open neighborhoods of $(s_o, \ldots, s_n, \ldots) \in \Sigma_o^*$.

3. We are left to showing that S is dense in $\hat{S}$. Let $b \in B$ and let

$$w(o,b) = [s_o, \ldots, s_n, \ldots].$$

Part 2 of this proof implies that $(V_n(s_n))_{n \geq o}$ is a basis of open neighborhoods of b. Let $n \geq o$ be arbitrarily given. Then $s_n \in V_n(s_n)$, i.e.

Sem. on Probab.
Vol.56 1988年
P1-120

$S \cap V_n(s_n) \neq \emptyset$ for all $n \geq o$. Hence $B \subset \overline{S}$ and by definition of $\hat{S} = B \cup S$, $\overline{S} = \hat{S}$.

4. From Proposition 3.9 we know that $\mathfrak{F}^*$ is a homeomorphism from $\hat{S}$ onto Σ_o^*. Moreover, Propositions 3.4 and 3.6 imply that also $\mathfrak{F}$ is a homeomorphism. This completes the proof of the theorem. _|

Sem. on Probab.
Vol.56 1988年
P1-120

§ 4 The Martin representation of nonnegative harmonic functions

The aim of this section is to establish the fact that the space $\hat{S}$ attached to a given infinite, locally finite tree S is a *Martin compactification* Y in the sense of the following two properties:

(M 1) Y is a compact, metrizable space which contains S as a dense open subspace.

(M 2) Y is a $\mathcal{C}$-compactification of S for the system $\mathcal{C}$ of continuous functions

$$t \to \frac{G(s,t)}{G(o,t)}$$

on S $(s \in S)$ in the sense that each such function extends to Y and the system of extensions separates the points of Y.

4.1 Proposition. Let $s \in S$ and $c([o,s]) =: [s_o,\dots,s_n]$. Then the sets

$$\hat{S}_j := \{x \in \hat{S} : s_j \in w(s,x),\ s_j \in w(o,x)\}$$

$(j=o,\dots,n)$ are open and constitute a partition of $\hat{S}$; $t \in S$ belongs to $\hat{S}_j$ iff $d(s_j,t)<d(s_k,t)$ for all $k=o,\dots,n$ with $k \neq j$. For $t \in \hat{S}_j$ we have

$$\frac{G(s,t)}{G(o,t)} = \frac{G(s,s_j)}{G(o,s_j)}$$

Proof. Put $a_i := [s_{i-1},s_i]$ for all $i=1,\dots,n$. Then

$$\hat{S}_j = \{x \in \hat{S} : \text{The arcs } a_1,\dots,a_j,\ a_{j+1}^-,\dots,a_n^- \text{ point in the direction of } x\}$$

$(j=o,\dots,n)$.

Sem. on Probab.
Vol.56 1988年
P1-120

1. $\{\hat{S}_o,\dots,\hat{S}_n\}$ is a partition of $\hat{S}$.

Let $x \in \hat{S}$.

1st case: $x \in S$. Given s and o there exists, by Properties 1.9.4, a $t \in S$ such that

$$c(s,o) = c(s,t) \cdot c(t,o) ,$$

$$c(s,x) = c(s,t) \cdot c(t,x) ,$$

$$c(o,x) = c(o,t) \cdot c(t,x) ,$$

and these geodesics have t as their only common vertex. Thus there is exactly one $j \in \{o,\dots,n\}$ such that $t = s_j$ and $x \in \hat{S}_j$.

2nd case: $x \in B$. Then Theorem 1.14 shows that there exists exactly one vertex s_j of $c(s,o)$ which is common to $w(o,x)$ and $w(s,x)$. This means that $x \in \hat{S}_j$.

2. The sets $\hat{S}_j$ are open in $\hat{S}$ $(j = o,\dots,n)$.

Let $x \in \hat{S}_j$. Then the arcs $a_1,\dots,a_j,\bar{a}_{j+1},\dots,\bar{a}_n$ point in the direction of x . Putting

$$U := \left(\bigcap_{i=1}^{j} U(s_{i-1},s_i)\right) \cap \left(\bigcap_{i=j+1}^{n} U(s_i,s_{i-1})\right)$$

we see that $x \in U \subset \hat{S}_j$, and thus $\hat{S}_j$ is open in $\hat{S}$ by the very definition of $\hat{\sigma}$.

3. Let $t \in S \cap \hat{S}_j$ for $j \in \{o,\dots,n\}$. By construction of $\hat{S}_j$ we obtain that

$$d(s_j,t) < d(s_k,t) \quad \text{for all} \quad k = o,\dots,n, \quad k \neq j .$$

Applying Theorem 2.10 we get for s, s_j and t the equation

$$G(s,t) = G(s,s_j) \cdot G(s_j,s_j)^{-1} \cdot G(s_j,t) ,$$

and for o, s_j and t,

$$G(o,t) = G(o,s_j)\cdot G(s_j,s_j)^{-1}\cdot G(s_j,t)$$

which obviously implies the assertion. _|

4.2 <u>Proposition</u>. For every $s\in S$ there exists exactly one continuous, strictly positive function $K_{s,o}$ on $\hat{S}$ with the property that

$$\mathrm{Res}_S\, K_{s,o} = \frac{G(s,\cdot)}{G(o,\cdot)}.$$

<u>Proof.</u> We define $K_{s,o}$ by

$$K_{s,o}(x) := \frac{G(s,s_j)}{G(o,s_j)}$$

for all $x\in\hat{S}_j$, $j=o,\ldots,n$. From Proposition 4.1 we infer that $K_{s,o}$ is in fact well-defined. Then, again by Proposition 4.1,

$$K_{s,o}(t) = \frac{G(s,t)}{G(o,t)}$$

for all $t\in S$. Moreover $K_{s,o}(x)>o$ by 2.7.4. Clearly $\mathrm{Res}_{\hat{S}_j}\, K_{s,o}$ is constant for all $j=o,\ldots,n$ by definition and continuous on $\hat{S}$ by Proposition 4.1.

The uniqueness of $K_{s,o}$ is clear, since continuous functions which agree on the dense subset S of $\hat{S}$ are necessarily equal. One just notes that the set of points where two continuous functions with values in a Hausdorff space are equal is closed. _|

Sem. on Probab.
Vol.56 1988年
P1-120

Proposition 4.2 enables us to make the following

4.3 Definition. The positive function K on $S\times\hat{S}$ defined by

$$K(s,x) := K_{s,o}(x)$$

for all $(s,x) \in S\times\hat{S}$ is called the *Martin kernel* (w.r.t. o).

4.4 Properties.

4.4.1 $K(o,x) = 1$ for all $x\in\hat{S}$.

4.4.2 For every $s\in S$ the function $K(s,\cdot)$ is continuous.

4.4.3 For all $s,x\in S$

$$K(s,x) = \frac{G(s,x)}{G(o,x)}.$$

4.4.4 For all $b\in B$ the function $K(\cdot,b)$ is harmonic on S.

Proofs. Properties 4.4.2 and 4.4.3 follow directly from Proposition 4.2.

Next we observe that

$$K(o,x) = K_{o,o}(x) = G(o,x)\cdot G(o,x)^{-1} = 1$$

whenever $x\in S$. From $\overline{S} = \hat{S}$ together with Property 4.4.2 we obtain that

$$K(o,x) = 1$$

for all $x\in\hat{S}$, which proves Property 4.4.1.

It remains to establish Property 4.4.4. Let $b\in B$, $s\in S$ and $c(o,s) = [s_o,\ldots,s_n]$. We have to show that

$$K(s,b) = \sum_{\{t\in S:d(s,t)=1\}} N(s,t)\cdot K(t,b)$$

holds. By Proposition 4.1 there exists a $j\in\{o,\ldots,n\}$ with $b\in\hat{S}_j$, and by the proof of Proposition 4.2 we get

$$K(s,b) = \frac{G(s,s_j)}{G(o,s_j)} = \frac{G(s,x)}{G(o,x)}$$

for some $x\in\hat{S}_j\cap S$ with $d(s_j,x)\geq 2$. Thus $d(x,s)\geq 2$ by definition of $\hat{S}_j$ and

$$K(t,b) = \frac{G(t,x)}{G(o,x)}$$

for all $t\in S$ with $d(s,t) = 1$. We now apply 2.7.4 in order to get

$$G(s,x) = N.G(s,x) = \sum_{t\in S} N(s,t).G(t,x),$$

hence

$$\begin{aligned} K(s,b) &= G(s,x).G(o,x)^{-1} \\ &= \sum_{t\in S} N(s,t).G(t,x).G(o,x)^{-1} \\ &= \sum_{\{t\in S:d(s,t)=1\}} N(s,t).G(t,x).G(o,x)^{-1} \\ &= \sum_{\{t\in S:d(s,t)=1\}} N(s,t).K(t,b), \end{aligned}$$

as was to be shown. _|

4.5 Proposition. Let $C\subset\Gamma$ have the following property: For no $c\in C$ there is a $c'\in\Gamma\setminus\Gamma(o)$ such that $c.c'\in C$.

Then for any superharmonic function $h\in \mathfrak{F}(S,\mathbb{R}_+)$,

$$U_C h\leq h.$$

Proof. For every $n\geq o$ we introduce the sets

$$C_n := C \cap (\bigcup_{j=o}^{n} \Gamma(j))$$

and

$$C'_n := C_n \cup \{c\in\Gamma(n)\text{: There exists a } c'\in\Gamma\setminus\Gamma(o) \text{ with } c.c'\in C\}$$

as well as

$$U_n := U_{C_n}$$

and

$$U'_n := U_{C'_n}.$$

We also set

$$T := \{s\in S\text{: There exists a } c\in C \text{ with } \alpha(c) = s\}.$$

Since

$$U'_o(s,t) := \begin{cases} 1 & \text{if } s=t \text{ and } [s]\in C'_o \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } s=t \text{ and } s\in T \\ 0 & \text{otherwise,} \end{cases}$$

Sem. on Probab.
Vol.56 1988年
P1-120

we obtain that

$$U_o'h(s) = \sum_{t\in S} U_o'(s,t)\cdot h(t) = \begin{cases} h(s) & \text{if } s\in T \\ o & \text{otherwise.} \end{cases}$$

Consequently

(1) $U_o'h \leq h$,

since h was assumed to be nonnegative.

Let $n\in\mathbb{N}$. For all $c\in\Gamma$ we put

$$q(c) := p(c)\cdot h(t),$$

where $t:=\beta(c)$. Then for every subset $\mathfrak{A}$ of $\{[t,s]:d(s,t)=1\}$ (where $t := \beta(c)$ is fixed) we obtain the following chain of inequalities:

$$\sum_{a\in\mathfrak{A}} q(c\cdot a) = \sum_{a\in\mathfrak{A}} p(c\cdot a)h(\beta(a))$$

$$= p(c) \sum_{a\in\mathfrak{A}} p(a)\cdot h(\beta(a)) \qquad \text{(by 2.2)}$$

$$= p(c) \sum_{\{s\in S:d(t,s)=1 \text{ and } [t,s]\in\mathfrak{A}\}} p([t,s])\cdot h(s)$$

(2)

$$\leq p(c) \sum_{\{s\in S:d(t,s)=1\}} p([t,s])\cdot h(s)$$

$$= p(c)\cdot Nh(t)$$

$$\leq p(c)\cdot h(t)$$

$$= q(c) \qquad \text{(since h is superharmonic).}$$

Now let $c\in C'_{n+1}\setminus C_n$, i.e. $\ell(c) = n+1$ and either $c\in C$ or $c\notin C$ but there exists a c' with $c\cdot c'\in C$. In the first case c can be written as $c'\cdot a$ with $\ell(a) = 1$ and $c'\notin C$ by assumption on C. The definition of C'_n yields $c'\in C'_n\setminus C_n$. In the second case $c\in C'_{n+1}\setminus C_{n+1}$, and by definition of C'_{n+1} and C'_n there are $c''\in C'_n\setminus C_n$ and $a\in\Gamma(1)$ such that $c = c''.a$. The discussion of both cases shows that every path $c\in C'_{n+1}\setminus C_n$ can be represented as $c'\cdot a$ with $c'\in C'_n\setminus C_n$ and $a\in\Gamma(1)$.

For each $s\in S$ we put

$$C(s) := \{c\in C : \alpha(c) = s\}.$$

Then (2) implies that

$$(3)\qquad \sum_{c\in C(s)\cap(C'_{n+1}\setminus C_n)} q(c) \leqq \sum_{c\in C(s)\cap(C'_n\setminus C_n)} q(c)$$

holds for all $s\in S$.

We also have

$$\begin{aligned} U'_n h(s) &= \sum_{t\in S} U'_n(s,t)\cdot h(t) \\ &= \sum_{t\in S}\sum_{c\in(C'_n)_{s,t}} p(c)\cdot h(t) \\ &= \sum_{t\in S}\sum_{c\in(C'_n)_{s,t}} q(c) \\ &= \sum_{c\in C'_n\cap C(s)} q(c) \end{aligned}$$

valid for all $s\in S$, which together with (3) implies

$$\begin{aligned} U'_{n+1}h(s) &= \sum_{c\in C'_{n+1}\cap C(s)} q(c) \\ &= \sum_{c\in C_n\cap C(s)} q(c) + \sum_{c\in(C'_{n+1}\setminus C_n)\cap C(s)} q(c) \end{aligned}$$

$$(4) \qquad \leq \sum_{c\in C_n \cap C(s)} q(c) + \sum_{c\in (C'_n \setminus C_n)\cap C(s)} q(c)$$

$$= \sum_{c\in C'_n \cap C(s)} q(c)$$

$$= U'_n h(s)$$

whenever $n \geq o$. Moreover, for every $n \geq o$ and $s \in S$ we have

$$U_n h(s) = \sum_{t\in S} \sum_{c\in (C_n)_{s,t}} p(c)\cdot h(t)$$

$$(5) \qquad \leq \sum_{t\in S} \sum_{c\in (C'_n)_{s,t}} p(c)\cdot h(t)$$

$$= U'_n h(s),$$

since $C_n \subset C'_n$ and $h \geq o$, and also

$$(6) \quad U_n h(s) \leq U_{n+1} h(s),$$

since $C_n \subset C_{n+1}$ and $h \geq o$.

Taking all information available together we obtain

$$\begin{array}{ccccccccccc} h & \underset{(1)}{\geq} & U'_o h & \underset{(4)}{\geq} & U'_1 h & \geq & \cdots & \geq & U'_n h & \underset{(4)}{\geq} & U'_{n+1} h \geq \cdots \\ & & \text{\rotatebox{90}{$\leq$}} \; (5) & & & & & & & & \\ & & U_o h & \underset{(6)}{\leq} & U_1 h & \leq & \cdots & \leq & U_n h & \underset{(6)}{\leq} & U_{n+1} h \leq \cdots . \end{array}$$

From $C_n \uparrow C$ as $n \to \infty$ we conclude that

$$U_c h = \lim_{n\to\infty} U_n h.$$

The sequence $(U'_n h(s))_{n \geq o}$ is decreasing and bounded, whence

$$\lim_{n\to\infty} U'_n h(s)$$

exists for all $s\in S$. And since $U_n'h \geqq U_n h$ for all $n\geq o$, the final assertion follows from

$$\begin{aligned} h(s) &\underset{(1)}{\geqq} U_o'h(s) \\ &\geqq \lim_{n\to\infty} U_n'h(s) \\ &\geqq \lim_{n\to\infty} U_n h(s) \\ &= U_C h(s) \end{aligned}$$

where $s\in S$. _|

4.6 Proposition. Let h be a superharmonic function $\in \mathfrak{F}(S,\mathbb{R}_+)$ and let T be a finite subset of S. Then there exists exactly one function $v\in \mathfrak{F}(S,\mathbb{R}_+)$ such that

$$\mathrm{Res}_{S\setminus T}\, v = o$$

and

$$\mathrm{Res}_T\, Gv = \mathrm{Res}_T h.$$

Proof. We consider the sets

$$\Gamma(T) := \{c\in\Gamma : \alpha(c),\ \beta(c)\in T\},$$

$$\Pi(T) := \{c=[s_o,\dots,s_n]\in\Gamma(T)\setminus\Gamma(o) : \text{no intermediate vertex of } c \text{ belongs to } T\}$$

and

$$[T] := \{[s]\in\Gamma(o) : s\in T\}$$

as well as the corresponding kernels $G_T := U_{\Gamma(T)}$, $N_T := U_{\Pi(T)}$ and $I_T := U_{[T]}$. By definition of the kernels G_T and I_T we get

$$G_T(s,t) = \begin{cases} G(s,t) & \text{if } (s,t)\in T\times T \\ o & \text{otherwise} \end{cases}$$

Sem. on Probab.
Vol.56 1988年
P1-120

and

$$I_T(s,t) = \begin{cases} I(s,t) & \text{if } (s,t)\in T\times T \\ o & \text{otherwise.} \end{cases}$$

Every path $c\in\Gamma(T)\setminus[T]$ can be represented as the unique product $c=c'\cdot c''$ with $c'\in\Gamma(T)$ and $c''\in\Pi(T)$; one just chooses $\beta(c') = \alpha(c'')$ as that vertex of c in which c hits T for the last but one time. Conversely, every path of this form belongs to $\Gamma(T)\setminus[T]$. Now we infer from 2.6 that

$$G_T = U_{\Gamma(T)\setminus[T]} + U_{[T]}$$

(1) $$= G_T\cdot N_T + I_T$$

Putting $v := I_Th - N_Th$ we obtain a function v von S satisfying

$$v(s) = \begin{cases} h(s) - N_Th(s) & \text{if } s\in T \\ o & \text{otherweise.} \end{cases}$$

Here one notes that $N_Th<\infty$.

a) We show that v is nonnegative.

$\Pi(T)$ does not contain two different paths of the form c and $c\cdot c'$, since otherwise there would exist an intermediate point in T. But then Proposition 4.5 implies $N_Th\leq h$ and hence $v(s) = h(s) - N_Th(s)\geq o$ for all $s\in T$.

b) From $I_Th = v+N_Th$ and $G_T = G_T\cdot I_T$ we conclude that

$$G_Th = G_T\cdot I_Th = G_T\ (I_Th)$$

(2) $$= G_T\ (v+N_Th)$$

$$= G_Tv+G_T\cdot N_Th$$

Sem. on Probab.
Vol.56 1988年
P1-120

But by (1) we have

(3) $G_T h = I_T h + G_T \cdot N_T h.$

T is finite and $\mathrm{Res}_{\complement(T\times T)}\, G_T = o$, whence $G_T h(s) = o$ for all $s \in S \setminus T$ and for $s \in T$

$$G_T h(s) = \sum_{t\in S} G_T(s,t)\cdot h(t)$$

$$= \sum_{t\in T} G_T(s,t)\cdot h(t)$$

$$= \sum_{t\in T} G(s,t)\cdot h(t) < \infty$$

We note that the functions on the right sides of the equations (2) and (3) are also finite (and nonnegative). Subtraction of (3) from (2) yields

(4) $G_T v = I_T h.$

Finally we get for all $s \in T$

$$Gv(s) = \sum_{t\in S} G(s,t)\cdot v(t)$$

$$= \sum_{t\in T} G(s,t)\cdot v(t)$$

$$= G_T v(s)$$

and therefore

$$h(s) = I_T h(s) = G_T v(s) = Gv(s).$$

c) The uniqueness of v follows from the subsequent Lemma. In fact every function w on S having the properties of v is necessarily of the form

Sem. on Probab.
Vol.56 1988年
P1-120

$$w = I_T h - N_T h. \quad \_|$$

4.7 Lemma. The function v is uniquely determined by the two properties

(a) $v(s) = o$ for $s \in S \setminus T$ and

(b) $Gv(s) = h(s)$ for $s \in T$

provided T is finite (and h arbitrary).

Proof. For $s \in T$ we have $h(s) = I_T h(s)$ and $Gv(s) = G_T v(s)$. Hence $I_T h = G_T v$. Since

$$(I_T - N_T).G_T = G_T - N_T.G_T = I_T,$$

which follows from $I.G_T = G_T$ and $G_T = I_T + N_T.G_T$, we obtain for $s \in T$ that

$$v(s) = I_T v(s) = (I_T - N_T).G_T v(s)$$

$$= (I_T - N_T).I_T h(s)$$

$$= I_T h(s) - N_T h(s).$$

Note that, conversely, the function $v := I_T h - N_T h$ satisfies properties (a) and (b). _|

4.8 Preparation: Measures on Σ_o.

In 3.3 we introduced the space

$$\Sigma_o := \lim_{\rightarrow} S_n,$$

Sem. on Probab.
Vol.56 1988年
P1-120

and in Theorem 3.10 we showed that the closed subspace B of $\hat{S}$ is homeomorphic to Σ_o.

4.8.1 The topology induced from $\hat{S}$ on B is generated by the system

$$\{B_{n,s},\ B\setminus B_{n,s}\ :\ n\in\mathbb{Z}_+,\ s\in S_n\}$$

of sets

$$B_{n,s} := \{b\in B: w(o,b)\ni s\}$$

and their complements in B, as follows from Proposition 3.8.

4.8.2 The topology of Σ_o ist generated by the system

$$\{A_{n,s}\ :\ n\in\mathbb{Z}_+,\ s\in S_n\}$$

of sets

$$A_{n,s} := \left(\prod_{m=o}^{n-1} S_m\times\{s\}\times\prod_{m>n} S_m\right)\cap\Sigma_o.$$

4.8.3 For $n\in\mathbb{Z}_+$, $s\in S_n$ and $c(o,s) := [s_o,\ldots,s_n]$ we obtain

$$A_{n,s} = (\{s_o\}\times\{s_1\}\times\cdots\times\{s_n\}\times\prod_{m>n} S_m)\cap\Sigma_o,$$

whence

$$A_{n,s}\cap A_{m,t}\in\{\emptyset,A_{n,s},A_{m,t}\}$$

whenever $n,m\in\mathbb{Z}_+$. Thus the system

$$\{A_{n,s}\ :\ n\in\mathbb{Z}_+,\ s\in S_n\}$$

Sem. on Probab.
Vol.56 1988年
P1-120

is a basis of the topology of Σ_o,

$$\tilde{\mathfrak{I}}(B_{n,s}) = A_{n,s}$$

for all $n \in \mathbb{Z}_+$, $s \in S_n$, and every open subset of Σ_o is a pairwise disjoint union of basic sets. This last fact implies that any (Borel) measure on Σ_o is determined by its values at basic sets.

4.8.4 Let $\tilde{\mu}$ be a measure on Σ_o. Then for every $n \in \mathbb{Z}_+$ we define a measure q_n on S_n by

$$q_n(s) := \tilde{\mu}\,(A_{n,s}) \text{ for } s \in S_n.$$

The sequence $(q_n)_{n \geq o}$ of measures q_n on S_n is *consistent* in the sense that the following property (C) holds:

For $n \in \mathbb{Z}_+$ and $s \in S_n$

$$\text{(C)} \quad q_n(s) = \sum_{\{s' \in S_{n+1} : d(s,s')=1\}} q_{n+1}(s') \quad ,$$

since
$$\begin{aligned} q_n(s) &= \pi_{n+1}\,(q_{n+1})(s) \\ &= q_{n+1}\,(\pi^{-1}_{n+1}(s)) \\ &= q_{n+1}\,(\bigcup_{\{s' \in S_{n+1} : d(s,s')=1\}} \{s'\}) . \end{aligned}$$

Moreover it is well-known that any measure on Σ_o is uniquely determined by a consistent sequence.

The relationship between measures μ on B and sequences $(q_n)_{n \geq o}$ can be formalized via the equalities

Sem. on Probab.
Vol.56 1988年
P1-120

$$\mu(B_{n,s}) = \mu(\mathfrak{I}^{-1}(A_{n,s}))$$
$$= \mathfrak{I}(\mu)(A_{n,s})$$
$$=: \tilde{\mu}(A_{n,s})$$
$$= q_n(s)$$

valid for all $n \in \mathbb{Z}_+$, $s \in S_n$.

We are now prepared for the proof of the integral representation of nonnegative harmonic functions.

4.9 Theorem. Let o be a fixed vertex of the tree S. For every non-negative harmonic function h on S there exists exactly one measure $\mu^h := \mu_o^h$ on B such that

$$h(s) = \int_B K(s,x)\ \mu^h(dx)$$

for all $s \in S$.

Proof. 1. Uniqueness. Let μ be a measure on B and let $(q_n)_{n \geq o}$ be the corresponding consistent sequence. We consider the function

$$s \to g(s) := \int_B K(s,x)\mu(dx)$$

on S. Let $n \in \mathbb{Z}_+$ and $s \in S(n)$. If $t \in S_n$ and $b \in B_{n,t}$, then $w(o,b)$ and $w(s,b)$ contain t. There exists exactly one vertex s_1 common to $c(o,t)$, $c(s,t)$ and $c(o,s)$. But then the definition of the kernel K implies that

$$K(s,b) = G(s,s_1)\cdot G(o,s_1)^{-1}$$
$$= G(s,t)\cdot G(o,t)^{-1}$$

which means that the function

Sem. on Probab.
Vol.56 1988年
P1-120

$$b \to K(s,b)$$

is constant on the set $B_{n,t}$ for all $s \in S(n)$, and thus

$$g(s) = \int_B K(s,x)\mu(dx)$$

$$= \sum_{t\in S_n} \int_{B_{n,t}} K(s,x)\mu(dx)$$

(1)

$$= \sum_{t\in S_n} G(s,t)\cdot G(o,t)^{-1}\mu(B_{n,t})$$

$$= \sum_{t\in S_n} G(s,t)\cdot q_n(t)\cdot G(o,t)^{-1}$$

for all $s \in S(n)$, the latter equality resulting from 4.8.4. Applying Proposition 4.6 to the set $T := S(n)$ we obtain the existence of exactly one function $v_n \in \mathfrak{F}(S,\mathbb{R}_+)$ such that

$$v_n(s) = o \quad \text{for all } s \notin S(n)$$

and

$$Gv_n(s) = h(s) \text{ for all } s \in S(n).$$

Since every vertex in $S(n-1)$ is joined only with vertices of $S(n)$, we get for all $s \in S(n-1)$,

$$- v_n(s) = N\cdot Gv_n(s) - Gv_n(s)$$

$$= N\cdot Gv_n(s) - h(s)$$

$$= \sum_{\{t\in S:d(s,t)=1\}} N(s,t)\cdot h(t) - h(s)$$

$$= \Delta h(s),$$

Sem. on Probab.
Vol.56 1988年
P1-120

and since h is assumed to be harmonic, we conclude that

$$\mathrm{Res}_{S\setminus S_n} v_n = o$$

which implies

$$Gv_n(s) = \sum_{t\in S} G(s,t)\cdot v_n(t)$$

$$= \sum_{t\in S_n} G(s,t)\cdot v_n(t)$$

or

(2) $$h(s) = \sum_{t\in S_n} G(s,t)\cdot v_n(t)$$

for all $s\in S(n)$.

Equations (1) and (2) show that the equality $g = h$ holds iff

$$\sum_{t\in S_n} G(s,t)\cdot v_n(t) = \sum_{t\in S_n} G(s,t)\left[q_n(t)\cdot G(o,t)^{-1}\right]$$

for all $s\in S(n)$ ($n\geq o$). Thus we have

(3) $$q_n(t) = v_n(t)\cdot G(o,t)$$

for all $n\geq o$ and $t\in S_n$. Since by Lemma 4.7 v_n is determined by its values on S_n, Preparation 4.8.4 assures that the representing measure μ on B is unique.

2. Existence. Reconsider the sequence $(q_n)_{n\geq o}$ of functions

$$t\to q_n(t) := G(o,t)v_n(t)$$

on S_n, $n\geq o$, introduced in part 1. of this proof. It suffices to show that this sequence $(q_n)_{n\geq o}$ is consistent in the sense that

Sem. on Probab.
Vol.56 1988年
P1-120

$$q_n(t) = \sum_{\{t'\in S_{n+1}:d(t,t')=1\}} q_{n+1}(t')$$

holds for all $t\in S_n$, $n\geq o$.

For $n\geq o$ and $t\in S_n$ we put

$$S_{n,t} := \{t'\in S_{n+1}:d(t,t') = 1\}$$

and for $n\geq o$ we define the function $q : S_n\to IR_+$ by

$$q(t) := \sum_{t'\in S_{n,t}} q_{n+1}(t')$$

for all $t\in S_n$. Then from Propositon 4.1 we see that

$$(4) \quad G(s,t')\cdot G(o,t')^{-1}=G(s,t)\cdot G(o,t)^{-1}$$

whenever $t\in S_n$, $s\in S(n)$ and $t'\in S_{n,t}$. But then we obtain for all $s\in S(n)$ the following chain of equalities:

$$\sum_{t\in S_n} G(s,t)(q(t)\cdot G(o,t)^{-1})$$

$$= \sum_{t\in S_n} G(s,t)(\sum_{t'\in S_{n,t}} q_{n+1}(t')G(o,t)^{-1})$$

$$= \sum_{t\in S_n}\sum_{t'\in S_{n,t}} q_{n+1}(t')\cdot G(s,t)\cdot G(o,t)^{-1}$$

$$= \sum_{t\in S_n}\sum_{t'\in S_{n,t}} q_{n+1}(t')\cdot G(s,t')\cdot G(o,t')^{-1}$$

$$= \sum_{t'\in S_{n+1}} G(s,t')\cdot q_{n+1}(t')\cdot G(o,t')^{-1} \quad (\text{since } \bigcup_{t\in S_n} S_{n,t} = S_{n+1})$$

Sem. on Probab.
Vol.56 1988年
P1-120

$$= \sum_{t' \in S_{n+1}} G(s,t') \cdot v_{n+1}(t')$$

$$= h(s) \quad \text{(by (2) of 1. and } S(n) \subset S(n+1),\ n \geq o)$$

$$= \sum_{t \in S_n} G(s,t) \cdot v_n(t) \quad \text{(by (2) of 1.)}$$

Thus by Lemma 4.7

$$q(t) \cdot G(o,t)^{-1} = v_n(t)$$

and so

$$q_n(t) = G(o,t) \cdot v_n(t) = q(t)$$

for all $t \in S_n$, $n \geq o$.

This means that the sequence $(q_n)_{n \geq o}$ is consistent, and by 4.8.4 there exists a corresponding measure μ^h on B. Then from (2) and (1) we conclude

$$h(s) = \sum_{t \in S_n} G(s,t) q_n(t) G(o,t)^{-1} = \int_B K(s,x) \mu^h(dx)$$

for all $s \in S$. $\_|$

4.10 Remark. The statement of the preceding theorem generalizes as follows: For every vertex s' of S there exists a function $K_{s,s'}$ on $\hat{S}$ satisfying a restriction condition similar to that in Proposition 4.2, an for every nonnegative harmonic function h on S there exists a unique measure $\mu^h_{s'}$ on B satisfying an integral representation analogous to that of Theorem 4.9.

4.11 Definition. For every measure μ on $\hat{S}$ the function

$$s \to K\mu(s) := \int_{\hat{S}} K(s,x) \mu(dx)$$

defined on S is called the *Martin potential* of μ.

Sem. on Probab.
Vol.56 1988年
P1-120

Theorem 4.9 together with the subsequent properties of Martin potentials will finally yield the fact announced at the beginning of the section that $\hat{S}$ is a $\mathcal{C}$-compactification.

4.12 Properties.

4.12.1 For every finite measure μ on $\hat{S}$, $K\mu$ is a nonnegative superharmonic function on S.

[Clearly $K\mu \geq o$. We put

$$v(s) := \frac{\mu(\{s\})}{G(o,s)}$$

for all $s \in S$, and $\nu := \mathrm{Res}_B \mu$. Since $\hat{S} = S \cup B$ we have for all $s \in S$ that

$$K\mu(s) = \int_S K(s,x)\mu(dx) + \int_B K(s,b)\nu(db)$$

$$(*) \qquad = \sum_{t \in S} K(s,t)\mu(\{t\}) + \int_B K(s,b)\nu(db)$$

$$= \sum_{t \in S} G(s,t)\cdot v(t) + \int_B K(s,b)\nu(db) \quad \text{(by Property 4.4.3)}$$

$$= Gv(s) + \int_B K(s,b)\nu(db).$$

For the function

$$s \to g(s) := \int_B K(s,b)\nu(db)$$

in $\mathfrak{F}(S,\mathbb{R}_+)$ we obtain via Property 4.4.4 that

$$Ng(s) = \sum_{\{t \in S: d(s,t)=1\}} N(s,t)\cdot g(t)$$

$$= \int_B \Big(\sum_{\{t \in S: d(s,t)=1\}} N(s,t)\cdot K(t,b)\Big)\nu(db)$$

Sem. on Probab.
Vol.56 1988年
P1-120

$$= \int_B K(s,b)\nu(db)$$

$$= g(s)$$

for all $s \in S$, which implies that g is harmonic. Thus by Theorem 2.12 $K\mu$ is superharmonic. ⌉

4.12.2 Conversely, if g is a nonnegative superharmonic function on S, then there exists exactly one measure μ on $\hat{S}$ such that $g = K\mu$.

[Let $g \in \mathfrak{F}(S, \mathbb{R}_+)$ be superharmonic. There exist a function $v \in \mathfrak{F}(S, \mathbb{R}_+)$ and a harmonic function $h \in \mathfrak{F}(S, \mathbb{R}_+)$ such that

$$g = Gv + h.$$

This is Theorem 2.12. We define a measure μ on $\hat{S}$, elementwise on S by

$$\mu(\{s\}) := v(s) \cdot G(o,s)$$

for all $s \in S$, and globally on B by

$$\mathrm{Res}_B\ \mu := \mu^h,$$

where h results from Theorem 4.9. Then from (*) we conclude that $g = K\mu$.

Concerning the uniqueness of μ we argue as follows:
Assume that $g = K\mu$ and define

$$\bar{v}(s) := \frac{\mu(\{s\})}{G(o,s)}.$$

Sem. on Probab.
Vol.56 1988年
P1-120

Then by (*)

$$g = G\overline{v} + \int_B K(.,b)\mu(db),$$

and the integral represents a harmonic function.

Furthermore,

$$\int_B K(.,b)\mu(db)<\infty,$$

$G\overline{v}<\infty$, and $\overline{v}<\infty$. Now Theorem 2.12 implies $v = \overline{v}$ and

$$h = \int_B K(.,b)\mu(db).$$

Hence, by Theorem 4.9, μ is uniquely determined on B. This also holds on S since $\mu(\{s\}) = G(o,s)\overline{v}(s)$. One just notes that by Theorem 2.12 v and h are uniquely determined by g. ⌉

4.12.3 Let the superharmonic function $g \in \mathfrak{F}(S,\mathbb{R}_+)$ admit the Riesz decomposition $g = Gv+h$. Then $g = K\mu$ iff μ is of the form given in Property 4.12.2. If $g=K\mu$, then $\mu(\hat{S}) = g(o)$.

[The first statement is clear; the second one follows by an application of Property 4.4.1 from

$$g(o) = K\mu(o) = \int_{\hat{S}} K(o,x)\mu(dx) = \mu(\hat{S}).]$$

4.12.4 Let g be a superharmonic function in $\mathfrak{F}(S,\mathbb{R}_+)$ represented as the Martin potential $g = K\mu$ for a unique measure μ on $\hat{S}$. Then g is harmonic iff $\mu(\hat{S}\setminus B) = o$.

[As was shown above,

$$g = Gv + \int_B K(.,b)\nu(db)$$

is the Riesz-decomposition of g. Hence g is harmonic iff $v = o$, i.e. $\mu(\{s\}) = v(s)G(o,s) = o$ for all $s \in S$. ⌉

Sem. on Probab.
Vol.56 1988年
P1-120

4.13 Theorem. $\hat{S}$ is a Martin compactification of S.

Proof. 1. That $\hat{S}$ is a compactification of S has been shown in Theorem 3.10. Thus (M1) of the definion at the beginning of the section is fulfilled.

2. In order that $\hat{S}$ satisfies (M2) of the definition of a Martin compactification it remains to be shown that the set $\{K(s,\cdot):s\in S\}$ of functions $K(s,\cdot)$ on $\hat{S}$ separates the points of $\hat{S}$. In fact, for a given $x\in\hat{S}$ the function $K(\cdot,x)$ on S is the Martin potential of the measure ε_x.

Now let $x,y\in\hat{S}$, $x\neq y$. Then $\varepsilon_x \neq \varepsilon_y$ and by Property 4.12.2, $K\varepsilon_x \neq K\varepsilon_y$ which means that there is an $s\in S$ such that $K(s,x) \neq K(s,y)$.

4.14 Notation. In view of the preceding theorem $\hat{S}$ is called *the Martin compactification of* S and the closed subspace $B := \hat{S}\setminus S$ of $\hat{S}$ is called *the Martin boundary of* S.

Sem. on Probab.
Vol.56 1988年
P1-120

§ 5 Convergence to the Martin boundary

This section is devoted to the probabilistic interpretation of the preceding discussion. Given the function p defined in § 2 on the set $\Gamma(1)$ of all arcs in a tree S, we will study the convergence to the Martin boundary of S of the Markov chain with state space S and transition matrix

$$(p([s,t]))_{s,t\in S}.$$

We shall strengthen General Assumption 2.2 by making the additional

5.1 Hypothesis that the above matrix is *stochastic* in the sense of the equality

$$\sum_{\{t\in S:d(s,t)=1\}} p([s,t]) = 1$$

valid for all $s\in S$.

5.2 Construction of the Markov chain

Consider the product set $S^{\mathbb{Z}_+}$, the product σ-algebra $\mathfrak{A}^{\mathbb{Z}_+}$ with $\mathfrak{A} := \mathfrak{P}(S)$ and for every $n\in\mathbb{Z}_+$, the n-th projection $Y_n := S^{\mathbb{Z}_+} \to S$. Then there exists a probability measure $\mathbb{P}$ on the measurable space $(S^{\mathbb{Z}_+}, \mathfrak{A}^{\mathbb{Z}_+})$ such that the sequence $(Y_n)_{n\in\mathbb{Z}_+}$ forms a *Markov chain* on the probability space $(S^{\mathbb{Z}_+}, \mathfrak{A}^{\mathbb{Z}_+}, \mathbb{P})$ with state space S, initial distribution ε_o and transition kernel P on S described by the *transition probabilities*

$$P(s,t) := \begin{cases} p([s,t]) & \text{if } d(s,t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$(s,t\in S)$.

Sem. on Probab.
Vol.56 1988年
P1-120

For this construction the fixed vertex o of S is the starting point of the chain, and P(s,t) is the probability that if the chain has arrived at time n in vertex s it will pass at time n+1 to the vertex t which is joined to s.

In order to determine the finite-dimensional distributions (with respect to $\mathbb{P}$) of $(Y_n)_{n\in\mathbb{Z}_+}$ we shall construct a stochastic process $(X_n)_{n\in\mathbb{Z}_+}$ on a probability space $(W,\mathfrak{F},\Pi)$ with state space S which is equivalent to $(Y_n)_{n\in\mathbb{Z}_+}$. $(X_n)_{n\in\mathbb{Z}_+}$ will also be a Markov chain with initial distribution ε_o and transition kernel P on S.

5.3 Notation. Let

W := set of all infinite paths in S.

Note that $W \subset S^{\mathbb{Z}_+}$.

For every $w = [s_o,\ldots,s_n,\ldots] \in W$ and $n \geq o$ we put

$$\rho_n(w) := [s_o,\ldots,s_n].$$

Let for $s\in S$

$$W^s := \{w\in W : \rho_o(w) = [s]\}$$

and let for $c \in \Gamma(n)$ with $n\geq o$

$$W(c) := \{w\in W : \rho_n(w) = c\}.$$

On W we define the σ-algebra $\mathfrak{F}$ generated by the system

$$\mathfrak{W} := \{W(c) : c\in\Gamma\}$$

(of cylinder sets).

5.4 Some properties.

5.4.1 W^o is identical with the set of trajectories (paths) of the Markov chain $(Y_n)_{n\in\mathbb{Z}_+}$.

5.4.2 If S carries the discrete topology and $W \subset S^{\mathbb{Z}_+}$ the topology $\mathcal{O}_W$ induced on W by the product topology on $S^{\mathbb{Z}_+}$, then $\mathfrak{F}$ is the Borel σ-algebra on W.

[Let $c = [s_o, \ldots, s_n] \in \Gamma(n)$ for some $n \geq o$. Then

$$W(c) = (\prod_{i=o}^{n} \{s_i\} \times \prod_{i>n} T_i) \cap W$$

with $T_i = S$ is open. We are left to show that the system $\mathfrak{W}$ of cylinder sets is a (countable) basis of $\mathcal{O}_W$. The sets

$$\prod_{i \geq o} O_i \cap W \quad \text{with } O_i \subset S \text{ for } i \geq o \text{ and } O_i = S \text{ for almost all } i \geq o$$

form a basis of $\mathcal{O}_W$ (by definition of the product topology). Let O be of the form $\prod_{i \geq o} O_i \cap W$ with $O_i = S$ for all $i>n$. Then

$$O = \bigcup_{c \in C} W(c)$$

with

$$C := \{c = [s_o, \ldots, s_m, \ldots, s_n] : s_m \in O_m \text{ for } o \leq m \leq n\}.$$]

5.4.3 The family $(W^s)_{s \in S}$ is a countable partition of W consisting of open sets.

[From

$$W^s = (\{s\} \times \prod_{i \geq 1} T_i) \cap W$$

with $T_i := S$ we see that W^s is open and closed. Since $\bigcup_{s \in S} W^s = W$, the assertion follows.]

Sem. on Probab.
Vol.56 1988年
P1-120

5.4.4 For every $s \in S$, W^s is a compact metrizable subspace of W.
[It is sufficient to show that W is a compact metric space. Define mappings $\xi(n+1):\Gamma(n+1)\to\Gamma(n)$ by

$$\xi(n+1)\ ([s_o,\dots,s_{n+1}]) := [s_o,\dots,s_n]$$

for all $[s_o,\dots,s_{n+1}]\in\Gamma(n+1)$. Then the corresponding projective limit $\varprojlim \Gamma(n)$ (with the discrete topology on $\Gamma(n)$) is a compact metric space, and the mapping

$$w \to (\rho_o(w),\ \rho_1(w),\dots,\rho_n(w),\dots)$$

from W^s into $\varprojlim \Gamma(n)$ is a bijection. Hence it remains to prove that ξ^{-1} is continuous, which follows from

$$\xi(\{s_o\}\times\dots\times\{s_n\}\times S\times\dots)$$
$$=(\{[s_o]\}\times\{[s_o,s_1]\}\times\dots\times\{[s_o,\dots,s_n]\}\times S\times\dots)\cap \varprojlim \Gamma(n).\ ]$$

Considering the projections $X_n : W \to S$ given as mappings

$$w := [s_o,\dots,s_n,\dots]\to s_n$$

we shall define a probability measure Π on $(W,\mathfrak{F})$ such that $(X_n)_{n\in\mathbb{Z}_+}$ becomes a Markov chain on $(W,\mathfrak{F},\Pi)$ with state space S which has the same finite-dimensional distributions (with respect to Π) as the chain $(Y_n)_{n\in\mathbb{Z}_+}$.

5.5 Proposition. On $(W,\mathfrak{F})$ there exists exactly one probability measure Π such that

$$\Pi(W(c)) = p(c)$$

for all $c\in\Gamma$ with $\alpha(c) = o$.

Moreover $\Pi(W\setminus W^o) = o$.

Sem. on Probab.
Vol.56 1988年
P1-120

Proof. For every $n \geq o$ we define mappings $q_n : \Gamma(n) \to \mathbb{R}_+$ by

$$q_n(c) := \begin{cases} p(c) & \text{if } \alpha(c) = o \\ o & \text{otherwise} \end{cases}$$

for all $c \in \Gamma(n)$, and $\xi_n : \Gamma(n+1) \to \Gamma(n)$ by

$$\xi_n([s_o,\ldots,s_n,s_{n+1}]) := [s_o,\ldots,s_n]$$

for all $[s_o,\ldots,s_n,s_{n+1}] \in \Gamma(n+1)$. Let $c \in \Gamma(n)$ with $\alpha(c) = o$ and $\beta(c) = t$. Then

$$\{c' \in \Gamma(n+1) : \xi_n(c') = c\}$$

$$= \{c' = c \cdot [t,u] : u \in S \text{ with } d(t,u) = 1\} =: C_c.$$

For $c' \in C_c$ we get

$$q_{n+1}(c') = p\,(c \cdot [t,u]) = p(c) \cdot p([t,u])$$

for some $u \in S$ with $d(t,u) = 1$. From Hypothesis 5.1 we infer that for $c \in \Gamma(n)$ with $\alpha(c) = o$ the following consistency condition holds:

$$\begin{aligned} q_n(c) &= p(c) \\ &= p(c) \sum_{\{u \in S : d(t,u)=1\}} p([t,u]) \\ &= \sum_{\{u \in S : d(t,u)=1\}} p(c) \cdot p([t,u]) \\ \text{(C)} \qquad &= \sum_{c' \in C_c} q_{n+1}(c') \end{aligned}$$

We note that (C) is trivially fulfilled for any $c \in \Gamma(n)$ with $\alpha(c) \neq o$.

Now consider the projective system

$$\Gamma(o) \overset{\xi_o}{\leftarrow} \Gamma(1) \leftarrow \cdots \leftarrow \Gamma(n) \overset{\xi_n}{\leftarrow} \Gamma(n+1) \leftarrow \cdots$$

as well as the mapping

$$w \to \rho(w) := (\rho_o(w), \rho_1(w), \ldots, \rho_n(w), \ldots)$$

Sem. on Probab.
Vol.56 1988年
P1-120

from W into $\lim\limits_{\leftarrow} \Gamma(n)$. ρ is clearly bijective.

For the given sequence $(q_n)_{n\geqq o}$ which is consistent in the sense of (C) there exists a unique probability measure Π on W such that

$$\Pi(W(c)) = \Pi\ (\rho_n^{-1}(c)) = q_n(c)$$

for all $n\geqq o$ and all $c\in\Gamma(n)$. This is the existence statement of the theorem.

Since Π satisfies the conditions

$$\Pi(W^o) = q_o([o]) = 1 \quad \text{(by General Assumption 2.2)}$$

and

$$\Pi(W^s) = q_o([s]) = 0$$

for all $s\in S$ with $s \neq o$, the support property of Π follows readily. _

5.6 Proposition. The Markov chains $(X_n)_{n\in\mathbb{Z}_+}$ and $(Y_n)_{n\in\mathbb{Z}_+}$ on $(W,\mathfrak{F},\Pi)$ and $(S^{\mathbb{Z}_+},\mathfrak{A}^{\mathbb{Z}_+},\mathbb{P})$ respectively have the same finite-dimensional distributions on S.

Proof. It suffices to show that for every $r\geqq 1$ the distributions of the random variables

$$\bigotimes_{n=1}^{r} X_n \quad \text{and} \quad \bigotimes_{n=1}^{r} Y_n$$

on W and $S^{\mathbb{Z}_+}$ respectively (with values in S^r) are identical.

For all $r\geqq 1$, $t_1,\ldots,t_r\in S$ we have to show that the equality

$$\mathbb{P}\,[Y_1 = t_1,\ldots,Y_r = t_r] = \Pi\ [X_1 = t_1,\ldots,X_r = t_r]$$

holds under the hypothesis $X_o = o$. Using the additional notation $t_o := o$ we get

Sem. on Probab.
Vol.56 1988年
P1-120

$$[X_1 = t_1, \ldots, X_r = t_r] = W([t_o, \ldots, t_r]).$$

By Proposition 5.5 combined with General Assumption 2.2 we obtain

$$\Pi(W([t_o, \ldots, t_r])) = p([t_o, \ldots, t_r]) = p([t_o, t_1]) \cdot \ldots \cdot p([t_{r-1}, t_r]),$$

whence

$$\Pi[X_1 = t_1, \ldots, X_r = t_r] = \Pi(W([t_o, \ldots, t_r])$$
$$= \prod_{i=1}^{r} p([t_{i-1}, t_i])$$
$$= \prod_{i=1}^{r} P(t_{i-1}, t_i).$$

Moreover we have that

$$P\,[Y_1 = t_1, \ldots, Y_r = t_r] = \prod_{i=1}^{r} P(t_{i-1}, t_i)$$

which is the defining property of the transition probabilities of the Markov chain $(Y_n)_{n \in \mathbb{Z}_+}$. From these last two equalities the assertion follows. _|

5.7 Proposition. Let $b \in B$ and $w(o,b) := [s_o, \ldots, s_n, \ldots]$. Then

$$\lim_{n \to \infty} s_n = b$$

in $\hat{S}$.

Sem. on Probab.
Vol.56 1988年
P1-120

Proof. By Part 2. of the proof of Theorem 3.10 the sequence $(V_n(s_n))_{n\geq o}$ is a neighborhood basis of b. Clearly the sequence $(s_n)_{n\geq o}$ in $\hat{S}$ converges towards b iff for every $n\geq o$ there is an $m\geq o$ such that $s_k \in V_n(s_n)$ for all $k\geq m$. Take $m := n$. Then for all $k\geq m$

$$[s_o,\ldots,s_n,\ldots,s_k] = c(o,s_k),$$

i.e. $c(o,s_k)$ contains s_n and hence $s_k \in V_n(s_n)$ for all $k\geq m = n$. ⌋

5.8 Proposition. Let $[t_o,\ldots,t_n,\ldots]$ be a path in W which contains any of its vertices only finitely many times.

(i) There exists a $b\in B$ such that

$$\lim_{n\to\infty} t_n = b \text{ (in } \hat{S}).$$

(ii) There exists a strictly increasing sequence $(k(n))_{n\geq o}$ in $\mathbb{Z}_+$ with $k(o) = o$ such that

$$w(t_o,b) = [s_o,\ldots,s_n,\ldots]$$

with $\quad s_n = t_{k(n)}$ for all $n\geq o$.

Proof. Without loss of generality we assume $t_o = o$. By assumption the set $\{k\in\mathbb{Z}_+ : t_k = s\}$ is finite for all $s\in S$. Hence the set

$$M_n := \{k\in\mathbb{Z}_+ : d(o,t_k)\leq n\}$$

is finite for all $n\geq o$. We introduce for all $n\geq o$

$$k(n) := \max\{k\in\mathbb{Z}_+ : k\in M_n\}$$

as well as

$$s_n := t_{k(n)}.$$

Sem. on Probab.
Vol.56 1988年
P1-120

Then $d(o,s_n) \leq n$ since $k(n) \in M_n$, and $d(o,t_{k(n)+1}) > n$ by definition of $k(n)$. Since $[t_o,\ldots,t_n,\ldots]$ is an infinite path in S, we have $d(s_n,t_{k(n)+1}) = 1$, hence

$$n < d(o,t_{k(n)+1}) \leq d(o,s_n) + d(s_n,t_{k(n)+1}) \leq n+1.$$

It follows that $d(o,s_n)=n$, i.e. $s_n \in S_n$ (for $n \geq o$).

From the definition of $k(n)$ we see that $d(o,t_k) > n$ for all $k > k(n)$. Moreover $c(o,t_k)$ is of the form

$$c_k := [s_o',\ldots,s_r']$$

for some $r > n$.

We will show by induction on $k > k(n)$ that the equality

$$(*) \qquad s_n' = s_n$$

holds: For $k=k(n)+1$ we deduce from $d(s_n,t_{k(n)+1})=1$, $s_n \in S_n$ and $t_{k(n)+1} \notin S_n$ that

$$c_{k(n)} = [s_o',\ldots,s_{r-1}']$$

and $s_{r-1}' = s_n$. Since $\ell(c_{k(n)})=n$, we get $r-1=n$ and $s_n'=s_n$. Now we assume that (*) has been proved for some $k>k(n)$. Since $d(t_{k+1},t_k)=1$, we have that either $c_{k+1} = [s_o',\ldots,s_{r-1}']$ or $c_{k+1} = [s_o',\ldots,s_r',t_{k+1}]$. This implies that c_{k+1} has the same vertex with index n as c_k, since $r>n$. By induction hypothesis (*) follows for $k+1$.

We observe that (*) implies that $t_k \in V_n(s_n)$ for all $k>k(n)$ $(n \geq o)$. In particular $s_{n+1} \in V_n(s_n)$ and therefore $d(s_n,s_{n+1})=1$ since

Sem. on Probab.
Vol.56 1988年
P1-120

$s_{n+1} \in S_{n+1}$, $s_n \in S_n$ for all $n \geq o$. Hence $[s_o,\ldots,s_n,\ldots]$ is an infinite chain in S with starting point o, i.e.

$$[s_o,\ldots,s_n,\ldots] \in \Sigma_o .$$

Theorem 3.10 tells us that $\mathfrak{F} : B \to \Sigma_o$ is a bijection, whence there exists $b \in B$ such that

$$w(o,b) = [s_o,\ldots,s_n,\ldots].$$

The sequence $(t_n)_{n\geq o}$ converges towards b, since $(V_n(s_n))_{n\geq o}$ is a basis of neighborhoods of b (by Part 2 of the proof of Theorem 3.10) and $t_k \in V_n(s_n)$ for all $k>k(n)$, $n\geq o$ (as was shown above). The proof of both statements (i) and (ii) is now complete. _|

The following result is the converse of the preceding proposition

5.9 Proposition. Let $w = [t_o,\ldots,t_n,\ldots] \in W$ be such that

$$\lim_{n\to\infty} X_n(w) = b$$

for some $b\in B$. Then w contains each of it vertices only finitely many times.

Proof. Suppose there exists a $t\in S$ with $t_i=t$ for infinitely many $i\geq o$, and let $b\in B$ be such that $\lim_{n\to\infty} t_n=b$ and $w(o,b) = [s_o,\ldots,s_n,\ldots]$. By Part 2. of the proof of Theorem 3.10 $(V_n(s_n))_{n\geq o}$ is a basis of neighborhoods of b. If we put $n := d(o,t)$ (which means that $t\in S_n$), then $t\notin V_{n+1}(s_{n+1})$. From $\lim_{n\to\infty} t_n = b$ we infer that $t_k \notin V_{n+1}(s_{n+1})$ only for finitely many $k\geq o$. Thus the above assumption is wrong, and the assertion follows. _|

Sem. on Probab.
Vol.56 1988年
P1-120

5.10 Notation. We set

$$W' := \{w \in W : \text{There exists } b \in B \text{ with } \lim_{n\to\infty} X_n(w) = b\},$$

and for all $w \in W'$

$$X(w) := \lim_{n\to\infty} X_n(w).$$

5.11 Technical Properties.

5.11.1 W' is a Borel subset of W.

[By Propostion 5.8, W' is the set of all infinite paths $w = [s_o, s_1, \ldots]$ with $\lim_{n\to\infty} d(o,s_n) = \infty$. Hence

$$W' = \bigcap_{n\in\mathbb{Z}_+} \bigcup_{m\in\mathbb{Z}_+} \bigcap_{k\geq m} \{w = [s_o, s_1, \ldots] \in W : d(o,s_k) > n\}$$

is a Borel set. One just notes that

$$\{w = [s_o, s_1, \ldots] \in W : d(o,s_k) > n\}$$
$$= W \cap (\prod_{i=1}^{k-1} T_i \times (S \setminus S(n)) \times \prod_{i>k} T_i)$$

with $T_i = S$.]

5.11.2 The mapping $X : W' \to B$ is a Borel mapping.

[For every closed subset A of B the limit of a convergent sequence (s_n) belongs to A iff $\lim_{n\to\infty} \hat{d}(s_n, A) = o$. Hence for closed A the set

$$X^{-1}(A) = \bigcap_{k\geq 1} \bigcup_{n\geq o} \bigcap_{m\geq n} X_m^{-1}(\{x \in S : \hat{d}(x,A) < \tfrac{1}{n}\})$$

is Borel. Consequently, X is a Borel mapping.]

5.12 Theorem. $\Pi(W \setminus W') = o$.

Or in probabilistic terms: $(X_n(w))_{n\in\mathbb{Z}_+}$ converges for Π- a.a. $w \in W$ towards an element of the Martin boundary B of S.

Proof. Let $t \in S$. For all $n \geq o$ we have

$$W^o \cap [X_n = t] = \{w \in W : X_o(w) = o,\ X_n(w) = t\}$$

$$= \bigcup_{c \in \Gamma(n)_{o,t}} W(c),$$

whence

$$\Pi[X_n = t] = \Pi\Big(\bigcup_{c \in \Gamma(n)_{o,t}} W(c)\Big)$$

$$= \sum_{c \in \Gamma(n)_{o,t}} \Pi(W(c))$$

$$= \sum_{c \in \Gamma(n)_{o,t}} p(c)$$

$$= P^n(o,t),$$

since $P^n = U_{\Gamma(n)}$ and Π is supported by W^o. Applying General Assumption 2.2 we obtain

$$\sum_{n \geq o} \Pi[X_n = t] = \sum_{n \geq o} P^n(o,t) = G(o,t) < \infty.$$

Hence the Borel-Cantelli Lemma implies that $\Pi(\bar{D}_t) = o$ for

$$\bar{D}_t := \limsup\, [X_n = t] = \bigcap_{n \geq 1} \bigcup_{m \geq n} [X_m = t].$$

We introduce the set

$$D_t := \{w \in W : X_n(w) = t \text{ for only finitely many } n \geq o\}.$$

Obviously, $\bar{D}_t = W \setminus D_t$ and from Propositons 5.8 and 5.9 we infer that $W' = \bigcap_{t \in S} D_t$. Since S is countable we thus obtain

$$\Pi(W \setminus W') = \Pi\Big(\bigcup_{t \in S} (W \setminus D_t)\Big) = o.$$

[In particular, from $W' = \bigcap_{t \in S} D_t$ follows again that W' is a Borel set.]

Sem. on Probab.
Vol.56 1988年
P1-120

5.13 Generalization. It is clear that Proposition 5.5 and Theorem 5.12 remain valid for any $s \in S$ instead of o. In this case there exists a unique probability measure Π_s on $(W, \mathfrak{F})$ satisfying

$$\Pi_s(W(c)) = p(c)$$

for all $c \in \Gamma$ with $\alpha(c) = s$, and

$$\Pi_s(W \setminus W^s) = o$$

as well as

$$\Pi_s(W \setminus W') = o.$$

Since Hypothesis 5.1 is equivalent to the assumption that the constant function h=1 on S is harmonic, the measure μ^1 of Theorem 4.8 is available. It turns out to be the limiting distribution of the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ as the following result shows.

5.14 Theorem. For all Borel subsets T of B

$$\Pi\,[X \in T] = \mu^1(T)$$

Proof. We start by introducing the measure $\mu := X(\Pi)$ on B. Recalling Preparation 4.7 and (3) of the proof of Theorem 4.8 it becomes clear that the measure μ^1 on B is uniquely determined by its values

(1) $\mu^1(B_{n,t}) = G(o,t) \cdot v_n(t)$

on the sets $B_{n,t}$ for $n \geq o$, $t \in S_n$. Here the functions v_n on S_n are given by the relationship

(2) $\sum_{t \in S_n} G(s,t) \cdot v_n(t) = 1$

valid for all $s \in S(n)$, $n \geq o$. We want to show that μ satisfies the equation (1) with μ^1 replaced by μ. For all $n \geq o$, $t \in S_n$ we define the Borel sets

Sem. on Probab.
Vol.56 1988年
P1-120

$$\Omega_{n,t} := [X \in B_{n,t}] = \{w \in W' : X(w) \in B_{n,t}\}$$

and

$$\Phi_{n,t} := \{w \in \Omega_{n,t} : \alpha(w)=t \text{ and } X_i(w) \neq t \text{ for all } i \geq 1\}.$$

Let

$$v_n(t) := \Pi_t(\Phi_{n,t}).$$

We will show that v_n satisfies equality (2) for $n \geq o$. Let $n \geq o$ and $t \in S_n$ as well as $s \in S(n)$ be given. We consider an infinite path $w \in \Omega_{n,t}$ with $\alpha(w)=s$. By Proposition 5.9 w contains each of its vertices only finitely many times. Hence by Proposition 5.8, w contains t. Thus $w = c.w'$ with $c \in \Gamma_{s,t}$ and $w' \in \phi_{n,t}$, and consequently

$$(3) \quad \Omega_{n,t} \cap W^s = \bigcup_{c \in \Gamma_{s,t}} c \cdot \Phi_{n,t},$$

where

$$c \cdot \Phi_{n,t} := \{c \cdot w' : w' \in \Phi_{n,t}\},$$

and the above is a countable union of pairwise disjoint sets. Since W^t is a Borel set by Property 5.4.3, every Borel set $\Phi \in W^t \cap \mathfrak{F}$ belongs to $\mathfrak{F}$. By definition of $\mathfrak{F}$ and by Property 5.4.2 we get for every Borel subset Φ of W^t and every path $c \in \Gamma_{s,t}$ that

$$(4) \quad \Pi_s(c \cdot \Phi) = p(c)\, \Pi_t(\Phi).$$

In fact it suffices to prove the validity of (4) only for

Sem. on Probab.
Vol.56 1988年
P1-120

cylinder sets $\Phi := W(c')$ with $c' \in \Gamma_{t,x}$ for some $x \in S$. But this is easy to see:

$$\begin{aligned}\Pi_s(c\cdot W(c')) &= \Pi_s(W(c\cdot c')) \\ &= p(c\cdot c') \\ &= p(c)\cdot p(c') \\ &= p(c)\ \Pi_t(W(c')).\end{aligned}$$

Next we get

$$\begin{aligned}\Pi_s(\Omega_{n,t}) &= \Pi_s(\Omega_{n,t}\cap W^s) \\ &= \Pi_s(\bigcup_{c\in\Gamma_{s,t}} c\cdot\Phi_{n,t}) \qquad \text{(by (3))} \\ &= \sum_{c\in\Gamma_{s,t}} \Pi_s\ (c\cdot\Phi_{n,t}) \\ &= \sum_{c\in\Gamma_{s,t}} p(c)\ \Pi_t(\Phi_{n,t}) \qquad \text{(by (4))} \\ &= \sum_{c\in\Gamma_{s,t}} p(c)\ v_n(t) \\ \text{(5)} \qquad &= G(s,t)\cdot v_n(t),\end{aligned}$$

the latter equality following from the definition of G.

For fixed $n\geq o$ the family $\{B_{n,t} : t \in S_n\}$ is a partition of B, hence the family $\{\Omega_{n,t} : t \in S_n\}$ is a partition of W'.
Thus for each $s \in S(n)$, $n\geq o$

$$\begin{aligned}1 &= \Pi_s(W) \\ &= \Pi_s(W') \qquad \text{(by Theorem 5.12)} \\ &= \Pi_s(\bigcup_{t\in S_n} \Omega_{n,t}) \\ &= \sum_{t\in S_n} \Pi_s(\Omega_{n,t}) \\ \text{(6)} \qquad &= \sum_{t\in S_n} G(s,t)\cdot v_n(t),\end{aligned}$$

Sem. on Probab.
Vol.56 1988年
P1-120

which shows that $v_n(t)$ satisfies equality (2).

Putting s := o in (5), we obtain that for all $n \geq o$, $t \in S_n$

$$\mu(B_{n,t}) = \Pi\left[X \in B_{n,t}\right]$$

$$= \Pi(\Omega_{n,t})$$

$$= G(o,t).v_n(t) \quad (7)$$

holds. Comparison of (7) and (6) with (1) and (2) respectively yields $\mu = \mu^1$, and the proof is **complete**. _|

Sem. on Probab.
Vol.56 1988年
P1-120

§ 6 Green and Martin kernel in the homogeneous case

As in the preceding section (S,A) denotes an infinite, locally finite tree. Under General Assumption 2.2 together with Hypothesis 5.1 we constructed a canonical Markov chain $(X_n)_{n\in\mathbb{Z}_+}$ on a probability space $(W, \mathfrak{F}, \Pi_s)$ with initial distribution ε_s and transition kernel $P = (P(s,t))_{s,t\in S}$ on the state space $(S, \mathfrak{P}(S))$. It turns out that General Assumption 2.2.2 can be interpreted as the transience of the Markov chain $(X_n)_{n\in\mathbb{Z}_+}$. In this section we shall drop this hypothesis and replace it by a homogeneity condition to be satisfied by the tree (S,A) and the transition kernel P of the given chain. We then show that under the homogeneity condition the Markov chain $(X_n)_{n\in\mathbb{Z}_+}$ is necessarily transient.

Before entering the discussion of the homogeneous case we shall give a

6.1 Probabilistic interpretation of the kernels G and F.

6.1.1 Clearly

$$G(s,t) = \sum_{c\in\Gamma_{s,t}} p(c)$$

$$= \sum_{n\geq o} P^n(s,t)$$

for all $s,t\in S$, and

$$G(s,t) < \infty$$

for all $s,t\in S$ is equivalent to General Assumption 2.2.2. Let

Sem. on Probab.
Vol.56 1988年
P1-120

$$\Phi := \{c=[s_o,\dots,s_n] : n\in\mathbb{N},\ s_i \neq s_n \text{ for all } o<i<n\}$$

$$= \bigcup_{n\in\mathbb{N}} \Phi(n)$$

with

$$\Phi(n) := \Phi \cap \Gamma(n)$$

for all $n\in\mathbb{N}$. Looking at the kernel F given by

$$F(s,t) := \sum_{c\in\Phi_{s,t}} p(c)$$

for all $s,t\in S$ one easily sees that

$$F(s,t) = \Pi_s([X_n=t \text{ für some } n\in\mathbb{N}]).$$

In fact, for all $n\in\mathbb{N}$, $s,t\in S$ we have

$$\sum_{c\in\Phi(n)_{s,t}} p(c) = \sum_{c\in\Phi(n)_{s,t}} \Pi_s(W(c))$$

$$= \Pi_s(\bigcup_{c\in\Phi(n)_{s,t}} W(c))$$

$$= \Pi_s([X_\nu \neq t \text{ for } o<\nu<n,\ X_n = t]),$$

and for all $s,t\in S$ we deduce from this that

$$F(s,t) = \sum_{c\in\Phi_{s,t}} p(c)$$

$$= \sum_{n>o} \sum_{c\in\Phi(n)_{s,t}} p(c)$$

$$= \sum_{n>o} \Pi_s([X_\nu \neq t,\ o<\nu<n,\ X_n = t])$$

$$= \Pi_s(\bigcup_{n>o} [X_\nu \neq t,\ o<\nu<n,\ X_n = t])$$

$$= \Pi_s([X_n = t \text{ for some } n\in\mathbb{N}]).$$

Sem. on Probab.
Vol.56 1988年
P1-120

6.2 Definition. A state $s \in S$ is said to be *transient* if

$$F(s,s) \quad 1,$$

and *recurrent* if

$$F(s,s) = 1.$$

Since $F(s,t)>o$ for all $s,t \in S$ there is a dichotomy in the sense that either all states of the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ are transient or all states are recurrent. In these two disjoint cases we call the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ transient or recurrent respectively.

From the general theory of Markov chains we know that a state $s \in S$ is transient iff $G(s,s)<\infty$.

In the following we drop General Assumption 2.2.2.

6.3 Proposition (Dichotomy). For the Green kernel G of the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ we have either $G(s,t)<\infty$ for all $s,t \in S$ or $G(s,t)=\infty$ for all $s,t \in S$.

Proof. Let $s,s', t \in S$ and $c_o \in \Gamma_{s,s'}$. By definition we have

$$G(s,t) = \sum_{c \in \Gamma_{s,t}} p(c)$$

and

$$G(s',t) = \sum_{c \in \Gamma_{s',t}} p(c).$$

Since the mapping $c \to c_o . c$ from $\Gamma_{s',t}$ into $\Gamma_{s,t}$ is injective and

$$p(c_o . c) = p(c_o) . p(c)$$

for all $c \in \Gamma_{s',t}$, we get

Sem. on Probab.
Vol.56 1988年
P1-120

$$G(s,t) = \sum_{c' \in \Gamma_{s,t}} p(c')$$

$$\geq \sum_{c \in \Gamma_{s',t}} p(c_o . c)$$

$$= p(c_o) \sum_{c \in \Gamma_{s',t}} p(c)$$

$$= p(c_o).G(s',t)$$

for all $c_o \in \Gamma_{s',s}$, and similarly

$$G(s,t) \geq p(c_1).G(s,t')$$

for all $c_1 \in \Gamma_{t',t}$ whenever $t' \in S$.

Now let $s,t \in S$ be such that $G(s,t) < \infty$. Then

$$G(s',t') \leq p(c_o)^{-1}.G(s,t')$$

for all $c_o \in \Gamma_{s,s'}$, whence

$$\leq p(c_o)^{-1} p(c_1)^{-1} G(s,t)$$

for all $c_1 \in \Gamma_{t',t}$, thus

$$< \infty$$

for all $s',t' \in S$.

Let conversely $s,t \in S$ be such that $G(s,t) = \infty$. Then

$$G(s',t') \geq p(c_o).G(s,t')$$

for all $c_o \in \Gamma_{s',s}$, whence

$$\geq p(c_o).p(c_1).G(s,t)$$

Sem. on Probab.
Vol.56 1988年
P1-120

for all $c_1 \in \Gamma_{t,t'}$, thus

$$= \infty$$

for all $s',t' \in S$. __|

6.4 Proposition. If on S there exists a superharmonic function $g \geq o$ which is not harmonic, then $G(s,t)<\infty$ for all $s,t \in S$.

Proof. Let $g \geq o$ be a superharmonic function such that $\Delta g<o$.
Putting

$$v := -\Delta g = g - Pg$$

we obtain that

$$v = g - Pg > o.$$

But then

$$Gv = \sum_{t \in S} G(.,t)v(t) \leq g$$

implies that $G(s,t)<\infty$ for all $s,t \in S$. __|

In place of General Assumption 2.2.2 we now make the following

6.5 Homogeneity Assumption. Let $q \in \mathbb{N}$, $q \geq 1$.

6.5.1 (S,A) is a *homogeneous tree of degree* q, i.e. for all $s \in S$

$$\text{card}(\{t \in S: d(s,t) = 1\}) = q+1.$$

6.5.2 $P(s,t) = \text{const}$ for all $s,t \in S$ such that $d(s,t) = 1$, i.e.

$$P(s,t) = \begin{cases} (q+1)^{-1} & \text{for all } s,t \in S \text{ with } d(s,t) = 1 \\ o & \text{otherwise}. \end{cases}$$

Sem. on Probab.
Vol.56 1988年
P1-120

6.6 Consequences.

6.6.1 For every $n \in \mathbb{Z}_+$ the values $P^n(s,t)$ of the transition kernels P^n depend only on $d(s,t)$. The same holds for the kernels G and F.

6.6.2 If G is finite, then

$$G(s,s) = \text{const} =: \alpha \quad \text{for all } s \in S$$

and

$$F(s,t) = \text{const} =: \beta \quad \text{for all } s,t \in S \text{ with } d(s,t) = 1.$$

Applying Propositions 2.8 and 2.9 we obtain for all $s,t \in S$ with $d(s,t) = n$ and $c(s,t) = [s_0, s_1, \ldots, s_{n-1}, s_n]$ that

$$\begin{aligned} G(s,t) &= I(s,t)+F(s,t).G(t,t) \\ &= \alpha F(s,t) \\ &= \alpha F(s,s_1).F(s_1,s_2)\cdot\ldots\cdot F(s_{n-1},t) \\ \text{(D)} \qquad &= \alpha\beta^n . \end{aligned}$$

We now assume that $q \geqq 2$.

6.7 Theorem. The Green kernel G is uniquely determined by the equality (D) of 6.6.2 and finite.

Proof. Suppose we have a finite kernel G' given by

$$G'(s,t) := \alpha'.\beta'^n$$

for all $s,t \in S$ such that $d(s,t)=n \in \mathbb{Z}_+$, where $\alpha',\beta' \in \mathbb{R}_+^\times$. Then for arbitrar $s,t \in S$ we get

$$G'.P(s,t) = \sum_{u \in S} G'(s,u)P(u,t)$$

$$= \sum_{\{u \in S: d(u,t)=1\}} G'(s,u)(q+1)^{-1}$$

Sem. on Probab.
Vol.56 1988年
P1-120

(i) If $s=t$, then $G'(s,u) = \alpha'.\beta'$ for all $u \in S$ with $d(s,u) = 1$ and hence $G'.P(s,t) = \alpha'.\beta'$.

(ii) If, however, $d(s,t) = n > o$ then by Observation 3.1 there exists exactly one vertex u with $d(u,t) = 1$ and $d(u,s) = n-1$. For all remaining vertices $v \neq u$ with $d(v,t) = 1$ we have $d(v,s) = n+1$. Consequently

$$G'.P(s,t) = (q+1)^{-1}(\alpha'.\beta'^{n-1} + q.\alpha'.\beta'^{n+1}).$$

Thus to $G' = I+G'.P$ there correspond the equations

(i') $\alpha' = 1+\alpha'.\beta'$

(ii') $\alpha'.\beta'^{n} = (q+1)^{-1}(\alpha'.\beta'^{n-1} + q.\alpha'.\beta'^{n+1})$ or equivalently $(q+1).\beta' = 1 + q.\beta'^{2}$.

Clearly, $\alpha' = q(q-1)^{-1}$ and $\beta' = q^{-1}$ are the unique solutions of the equations (i) and (ii). It follows that for all $n \in \mathbb{Z}_+$

$$\begin{aligned} G' &= I + (I+G'.P).P \\ &= I + P + G'.P^2 \\ &= \cdots \\ &= I + P + P^2 + \cdots + P^n + G'.P^{n+1} \\ &\geqq \sum_{k=o}^{n} P^k. \end{aligned}$$

Since this inequality holds for all $n \in \mathbb{Z}_+$ we obtain

$$G' \geqq G = \sum_{k \geqq o} P^k,$$

and the finiteness of G follows from that of G'. But the finiteness

Sem. on Probab.
Vol.56 1988年
P1-120

of G yields by 6.6.2 that G is uniquely determined by (D), thus $G' = G$. __|

6.8 Corollary. For all $s,t \in S$

$$G(s,t) = q^{1-d(s,t)}(q-1)^{-1}.$$

Proof. Let $s,t \in S$ such that $d(s,t) = n \in \mathbb{Z}_+$.

For n=o we have

$$\begin{aligned} G(s,s) &= I(s,s) + G.P(s,s) \\ &= 1 + q(q-1)^{-1}q^{-1} \text{ (by (i) of the proof of 6.7)} \\ &= 1 + (q-1)^{-1} \\ &= q(q-1)^{-1} . \end{aligned}$$

For n>o we get

$$\begin{aligned} G(s,t) &= I(s,t) + G.P(s,t) \\ &= q+1)^{-1}\left[q(q-1)^{-1}q^{1-n} + q^2(q-1)^{-1}q^{-1-n}\right] \\ &\qquad \text{(by (ii) of the proof of 6.7)} \\ &= q^{1-n}(q-1)^{-1}. \end{aligned}$$

In both cases the desired formula has been established. __|

6.9 Corollary. The given Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ is transient.

The proof is immediate by Corollary 6.8. __|

In order to compute the Martin kernel of a transient Markov chain on a homogeneous tree (S,A) of degree $q \geq 2$ we need the notion of a horocycle.

Sem. on Probab.
Vol.56 1988年
P1-120

From 3.7 we know that to each fixed end $b \in B$ and all $s,s' \in S$ there exists a $z \in \mathbb{Z}$ such that

$$d(s,t) - d(s',t) = z$$

for all vertices t common to $w(s,b)$ and $w(s',b)$. In fact, we just put

$$z := \delta_b(s,s').$$

6.10 Definition and properties of horocycles

6.10.1 For any $b \in B$ the relationship $\delta_b(s,s') = o$ between vertices $s,s' \in S$ is an equivalence. The corresponding classes are called the *horocycles associated with* b.

6.10.2 Fix the point $o \in S$. Then the nonempty ones among the sets

$$H_{n,b} := \{s \in S : \delta_b(s,o) = n\}$$

$(n \in \mathbb{Z})$ are horocycles associated with $b \in B$.

[Let H be an equivalence class with respect to the above equivalence relation. For $s \in H$ and $n := \delta_b(s,o)$ we obtain $s \in H_{n,b}$. But if $s' \in H$ is arbitrary, then by

$$\delta_b(s',o) = \delta_b(s',s) + \delta_b(s,o) = n$$

$s' \in H_{n,b}$. In both cases $H \subset H_{n,b}$.

Let, conversely, $H_{n,b} \neq \emptyset$ for some $n \in \mathbb{Z}$, and $s,t \in H_{n,b}$. Then

$$\begin{aligned}\delta_b(s,t) &= \delta_b(s,o) + \delta_b(o,t)\\ &= \delta_b(s,o) - \delta_b(t,o)\\ &= o,\end{aligned}$$

i.e. s and t are equivalent in the above sense.]

Sem. on Probab.
Vol.56 1988年
P1-120

6.10.3 Horocycles form a partition of S which is unique within a translation of the index n, and

$$\delta_b(s,s') = m-n$$

for all $s\in H_{m,b}$, $s'\in H_{n,b}$ with $m,n\in\mathbb{Z}$.

6.11 Application to the Martin kernel

Let $(H_{n,b})_{n\in\mathbb{Z}_+}$ be the family of normalized horocycles (with $o\in H_{o,b}$) associated with an end $b\in B$. In Proposition 4.1 we established a partition $\{\hat{S}_o,\dots,\hat{S}_n\}$ of the space $\hat{S} := S\cup B$. It was shown that for $b\in B$ there exists a $j\in\mathbb{N}$ with $o\leq j\leq n$ such that $b\in\hat{S}_j$. From this fact follows that

$$\begin{aligned} K(s,b) &= G(s,s_j).G(o,s_j)^{-1} \\ &= q^{1-d(s,s_j)}.q^{d(o,s_j)-1} \\ &= q^{d(o,s_j)-d(s,s_j)} \\ &= q^{\delta_b(o,s)} \\ &= q^{-\delta_b(s,o)} \\ &= q^{-n}. \end{aligned}$$

I.e. the mapping $s\to K(s,b)$ is constant on the horocycles $H_{n,b}$ associated with b, and

$$K(s,b) = q^{-n}$$

for all $s\in H_{n,b}$, $n\in\mathbb{Z}$.

Sem. on Probab.
Vol.56 1988年
P1-120

6.12 Theorem. For any function $h \geqq o$ on S the following statements are equivalent

(i) h is harmonic.

(ii) $\sum_{t \in S} P(s,t)h(t) = h(s)$ for all $s \in S$.

(iii) $\sum_{\{t \in S: d(s,t)=1\}} h(t) = h(s)(q+1)$ for all $s \in S$.

(iv) $h(s) = \int_B q^{-\delta_b(s,o)} \mu^h(db)$ for all $s \in S$.

The proof of the nontrivial equivalence (i) <==> (iv) makes the statement of Theorem 4.9 precise under the Homogeneity Assumption 6.5, with the help of 6.11. ⌋

6.13 Theorem. Let (S,A) be a homogeneous tree of degree $q \geqq 2$ and let $(X_n)_{n \in \mathbb{Z}_+}$ be a Markov chain with initial distribution ε_o and transition kernel $(P(s,t))_{s,t \in S}$ given by

$$P(s,t) := \begin{cases} (q+1)^{-1} & \text{if } d(s,t)=1 \\ o & \text{otherwise.} \end{cases}$$

Then

(i) $(X_n(w))_{n \in \mathbb{Z}_+}$ converges for Π-a.a. $w \in W$ towards $X(w) \in B$.

(ii) $X(\Pi)$ equals the measure μ^1 on B which is uniquely determined by the equality

$$\int_B q^{-\delta_b(s,o)} \mu^1(db) = 1$$

valid for all $s \in S$.

The proof is immediate from Theorems 4.9, 5.12 and 5.14 with the help of 6.11. ⌋

Sem. on Probab.
Vol.56 1988年
P1-120

§ 7 Transience of isotropic random walks

Let G be a unimodular locally compact group and K a compact subgroup of G. By ω_G we denote a fixed (left invariant) Haar measure of G. The pair (G,K) is said to be a *Gelfand pair* if the algebra $L^1(G//K)$ of K-biinvariant ω_G-integrable functions on G is commutative. The harmonic analysis of Gelfand pairs can be applied to probabilistic studies on trees. We shall illustrate this method by showing that a homogeneous tree (S,A) of degree $q \geqq 1$ can be viewed as a Gelfand pair and that isotropic random walks on (S,A) of degree $q \geqq 2$ are transient. The main portion of this section will be devoted to an expository presentation of the theory of Gelfand pairs applied to a homogeneous tree. The detailed discussion of the transience result will be essential also for the following section.

Let (S,A) be a homogeneous tree of degree $q \geqq 1$.

7.1 Definition. A bijective mapping $g : S \to S$ is said to be an *automorphism* of (S,A) if

$$\{g(s),\ g(t)\} \in A$$

for all $\{s,t\} \in A$.

Clearly, any automorphism g of (S,A) is an isometry in the sense that

$$d(s,t) = d(g(s),\ g(t))$$

for all $s,t \in S$.

Sem. on Probab.
Vol.56 1988年
P1-120

By Aut(S,A) we denote the set of all automorphisms of (S,A). A subset S' of S is called g-*invariant* for some $g \in Aut(S,A)$ if $g(S')=S'$.

7.2 Definition. Any sequence $(s_n)_{n\in Z}$ of vertices $s_n \in S$ with $d(s_n, s_{n+1})=1$ for all $n\in \mathbb{Z}$ and $s_n \neq s_m$ for $n\neq m$ $(m\in \mathbb{Z})$ is called a *doubly infinite chain*.

7.3 Theorem (J. Tits). Let $g\in Aut(S,A)$ and let

$$d := \min\{d(s,g(s)) : s\in S\}.$$

Then exactly one of the subsequent situations occurs:

(1) There is a g-invariant $s\in S$.

(2) There exist $s,t\in S$ with $d(s,t)=1$ such that

$$g(s) = t \text{ and}$$
$$g(t) = s.$$

(3) There exists a g-invariant doubly infinite chain $(s_n)_{n\in \mathbf{Z}_+}$ such that

$$g(s_n) = s_{n+k}$$

for all $n\in \mathbb{Z}$, $k\in \mathbb{N}$.

If (3) is satisfied, then $(s_n)_{n\in \mathbf{Z}}$ is uniquely determined, $k=d$, and the vertices of $(s_n)_{n\in \mathbf{Z}}$ are the only vertices $s\in S$ with $d(s,g(s))=d$.

7.4 Remark. $G := Aut(S,A)$ is a non-Abelian group with composition of mappings as operation and the identity mapping as neutral element $\mathbb{1}$. Clearly, to arbitrary vertices $s,t\in S$ there exists always a $g \in G$

Sem. on Probab.
Vol.56 1988年
P1-120

such that $g(s)=t$.

7.5 Topology in G. If one introduces in G the topology $\mathcal{O}_p$ of pointwise convergence, G becomes a unimodular locally compact group. Moreover, G is metrizable and separable with a metric d_G given as the mapping

$$(g,h) \to d_G(g,h) := \sum_{t\in S} d(g(t),h(t))(q+1)^{-2d(o,t)}$$

from $G\times G$ into $\mathbb{R}_+$.

7.6 Cosets of G. For the fixed vertex $o\in S$ we introduce the stabilizer

$$K := \{g\in G : g(o)=o\}$$

of o as a compact open subgroup of G.

7.6.1 G operates transitively and continuously on S under the operation $(g,s) \to g(s)$ from $G\times S$ into S. I.e. S is a homogeneous space.

7.6.2 $S \cong G/K$

In fact, there is a bijective and homeomorphic mapping ϕ from S onto G/K.

[By 7.6.1 the orbit of o equals the whole of S. For $t\in S$ there exists a $g \in G$ such that $t = g(o)$ and

$$\{h\in G : h(o)=t\} = gK.$$

If, conversely, $f\in G$. Then

$$fK = \{h\in G : h(o)=f(o)\}.$$

One defines $\phi : S \to G/K$ by $\phi(t) := gK$ where $g\in G$ and $g(o)=t$.]

Sem. on Probab.
Vol.56 1988年
P1-120

7.7 Double cosets of G. G operates on S×S under the operation $(g,t_1,t_2) \to (g(t_1), g(t_2))$ from G×S×S into S×S. We note that this operation is not transitive.

7.7.1 Given (s,s'), $(t,t') \in S\times S$ there exists a $g \in G$ such that $(g(s), g(s')) = (t,t')$ iff $d(t,t') = d(s,s')$.

7.7.2 For every $(s,s') \in S\times S$ the set

$$x(s,s') := \{(t,t') \in S\times S: \text{There exists a } g\in G:(g(s),g(s')) = (t,t')\}$$

is called the *orbit of* (s,s') *under* G. The relation

$$(s,s') \sim (t,t') :\iff x(s,s') = x(t,t')$$

is an equivalence relation on S×S. The set of all orbits under G will be abbreviated by X.

From 7.6.1 follows that the orbit $x_o := x(t,t)$ is independent of $t \in S$.

7.7.3 There is a one-to-one correspondence between the orbits $x(s,t)$ under G and the values $d(s,t)$ of d, i.e.

$$X \cong \mathbb{Z}_+ .$$

7.7.4 $\mathbb{Z}_+ \cong G//K$.

In fact, there is a bijective and homeomorphic mapping ϕ_1 from $\mathbb{Z}_+$ onto G//K.

[To $n \in \mathbb{Z}_+$ there exist $t \in S$ with $d(o,t)=n$ and $g \in G$ such that $g(o)=t$ (the latter by 7.6.1). Then

$$H := \{h\in G : d(o,h(o)) = d(o,g(o)) = n\} = KgK.$$

In fact, for $h \in H$ one chooses $g' \in G \cap K$ such that $g'(t') = t$ with

Sem. on Probab.
Vol.56 1988年
P1-120

$t' = h(o)$. Then $h = g'^{-1}gg^{-1}g'h \in KgK$. If, however, $f,h \in K$ then

$$d(o,fgh(o)) = d(o,fg(o)) = d(o,f(t)) = d(f(o), f(t))$$

$$= d(o,t) = n$$

which implies $fgh \in H$.

If, conversely, $f \in G$ then

$$KfK = \{h \in G : d(o,h(o)) = d(o,f(o))\}.$$

One defines $\phi_1 : \mathbb{Z}_+ \to G//K$ by $\phi_1(n) := KgK$ where $g \in G$ with $d(o,g(o)) = n$.]

7.8 Invariant measures. On G/K there exists a G-invariant positive measure σ' which is unique up to a multiplicative factor. σ' turns out to be the quotient measure ω_G/ω_K on G/K. But then $\sigma := \phi^{-1}(\sigma')$ is a positive measure on S. Since S and G/K are discrete spaces, σ can be chosen as the counting measure on S.

Let ϕ_2 denote the mapping $S \to \mathbb{Z}_+$ defined by $\phi_2(t) := d(o,t)$ for all $t \in S$. Then $\tau := \phi_2(\sigma)$ is a positive measure on $\mathbb{Z}_+$ satisfying

$$\tau(\{n\}) = \phi_2(\sigma)(\{n\})$$

$$= \sigma(\{t \in S : d(o,t) = n\})$$

$$= \sigma(S_n)$$

$$= (q+1)q^{n+1}$$

for all $n \in \mathbb{N}$ and

$$\tau(\{o\}) = 1 .$$

Sem. on Probab.
Vol.56 1988年
P1-120

7.9 The vector space $L^1(\mathbb{Z}_+,\tau)$ of complex sequences $\alpha := (\alpha_n)_{n\geq o}$ satisfying the condition

$$|\alpha_o| + \sum_{n\geq 1} (q+1)q^{n-1}|\alpha_n| < \infty$$

carries a norm defined for $f=(f[n])_{n\geq o} \in L^1(\mathbb{Z}_+,\pi)$ by

$$||f|| := |f[o]| + \sum_{n\geq 1} (q+1)q^{n-1}|f[n]|$$

and a convolution defined for $f=(f[n])_{n\geq o}$ and $g=(g[n])_{n\geq o}$ in $L^1(\mathbb{Z}_+,\tau)$ by

$$f*g[d(s,t)] := \int_S f[d(s,u)]g[d(u,t)]\sigma(du)$$
$$= \sum_{u\in S} f[d(s,u)]g[d(u,t)].$$

It is easily checked that $L^1(\mathbb{Z}_+,\tau)$ becomes a Banach algebra with unit element $e_o := (e_o[n])_{n\geq o}$ defined by

$$e_o[n] := \delta_{o,n}$$

for all $n\geq o$. Since S is a symmetric space in the sense of

$$x(s,t) = x(t,s)$$

for all $s,t\in S$, $L^1(G//K)\cong L^1(\mathbb{Z}_+,\tau)$ is commutative, hence (G,K) is a Gelfand pair.

From now on we shall adopt the terminology that (S,A) is considered as a Gelfand pair.

7.10 Spherical functions. Any non identically vanishing homomorphism χ from $L^1(\mathbb{Z}_+,\tau)$ into $\mathbb{C}$ is called a *character* of $L^1(\mathbb{Z}_+,\tau)$. Characters

Sem. on Probab.
Vol.56 1988年
P1-120

of $L^1(\mathbb{Z}_+,\tau)$ are continuous linear functionals of norm 1, and there is a bijection $\chi \to \chi^{-1}(o)$ from the set M of all characters of $L^1(\mathbb{Z}_+,\tau)$ onto the maximal ideal space Max $L^1(\mathbb{Z}_+,\tau)$ of $L^1(\mathbb{Z}_+,\tau)$. Moreover, to every $\chi \in M$ there exists exactly one $s = (s[n])_{n\geq o} \in L^\infty(\mathbb{Z}_+,\tau)$ such that χ has the form

$$\chi(f) = \sum_{n\geq o} f[n]s[n]\tau(\{n\})$$

for all $f := (f[n])_{n\geq o} \in L^1(\mathbb{Z}_+,\tau)$.

7.10.1 $s = (s[n])_{n\geq o} \in L^\infty(\mathbb{Z}_+,\tau)$ is said to be a *spherical function* on (G,K) (or (S,A)) if the mapping χ_s defined by

$$\chi_s(f) := \sum_{n\geq o} f[n]s[n]\tau(\{n\})$$

for all $f := (f[n])_{n\geq o} \in L^1(\mathbb{Z}_+,\tau)$ is a character of $L^1(\mathbb{Z}_+,\tau)$.

By S(G,K) we denote the totality of all spherical functions on (G,K).

7.10.2 $||s||_\infty := \sup|s[n]| = s[o] = 1$ for all $s \in S(G,K)$.

7.10.3 $s \in S(G,K)$ iff $s \in L^\infty(\mathbb{Z}_+,\tau)$ and

$$\sum_{n\geq o} (f*g)[n]s[n]\tau(\{n\})$$
$$= (\sum_{n\geq o} f[n]s[n]\tau(\{n\}))\ (\sum_{n\geq o} g[n]s[n]\tau(\{n\}))$$

for all $f,g \in L^1(\mathbb{Z}_+,\tau)$.

7.11 <u>Gelfand transform</u>. Here we discuss the relationship between the sets S(G,K) and $M = \{\chi_s : s \in S(G,K)\} \cong$ Max $L^1(\mathbb{Z}_+,\tau)$.

For any $f \in L^1(\mathbb{Z}_+,\tau)$ we define the *Gelfand transform* $\hat{f}$ of f by

$$\hat{f}(\chi_s) := \chi_s(f)$$

Sem. on Probab.
Vol.56 1988年
P1-120

The mapping $f \to \hat{f}$ from $L^1(\mathbb{Z}_+,\tau)$ onto $\hat{L}^1(\mathbb{Z}_+,\tau) = \{\hat{f} : f \in L^1(\mathbb{Z}_+,\tau)\}$ is said to be the Gelfand transformation. The weak topology induced by $\hat{L}^1(\mathbb{Z}_+,\tau)$ is called the *Gelfand topology*. Since $L^1(\mathbb{Z}_+,\tau)' \cong L^\infty(\mathbb{Z}_+,\tau)$, the mapping $s \to \chi_s$ from $S(G,K)$ onto M is an isometry, i.e. $M \cong S(G,K)$. Thus the Gelfand transform on M yields the *Fourier transform* on $S(G,K)$ in the sense that for a given $f \in L^1(\mathbb{Z}_+,\tau)$

$$\hat{f}(s) = \sum_{n \geq o} f[n] s[n] \tau(\{n\})$$

whenever $s \in S(G,K)$.

7.11.1 $S(G,K)$ is a compact metrizable space.

7.11.2 $(s_n)_{n \geq o} \in S(G,K)$ converges towards $s \in S(G,K)$ iff $(\hat{f}(s_n))_{n \geq o}$ converges towards $\hat{f}(s)$ for every $f \in L^1(\mathbb{Z}_+,\tau)$.

7.11.3 Given $f,g \in L^1(\mathbb{Z}_+,\tau)$ we have

$$\widehat{f*g}(s) = \hat{f}(s)\hat{g}(s)$$

for all $s \in S(G,K)$.

7.12 Spectral properties of the Gelfand transform. We introduce a sequence $(e_n)_{n \geq o}$ in $L^1(\mathbb{Z}_+,\tau)$ by

$$e_n[k] := \delta_{k,n}\, \tau(\{n\})^{-1}$$

for all $k \in \mathbb{Z}_+$. Clearly, e_o is the unit of the Banach algebra $L^1(\mathbb{Z}_+,\tau)$, $||e_n|| = 1$ and

$$e_n * e_1 = q(q+1)^{-1} e_{n+1} + (q+1)^{-1} e_{n-1}$$

for all $n \in \mathbb{Z}_+$. It follows that the functions e_n can be expressed as

Sem. on Probab.
Vol.56 1988年
P1-120

polynomials in e_1. Moreover, the set $\{P(e_1):P\in\mathbb{C}[x]\}$ is a dense subalgebra of $L^1(\mathbb{Z}_+,\tau)$.

7.12.1 For every $f\in L^1(\mathbb{Z}_+,\tau)$ an element z of $\mathbb{C}$ is called a *spectral value for* f if $f - z\, e_o$ has no inverse in $L^1(\mathbb{Z}_+,\tau)$.

The set $Sp(f)$ of spectral values for f is a nonempty compact subset of $\{z\in\mathbb{C}:|z|\leq\|f\|\}$.

7.12.2 The Gelfand transform of e_1 is a homeomorphism from M onto $Sp(e_1)$, thus the mapping $s\to\hat{e}_1(s)$ is a homeomorphism from $S(G,K)$ onto $Sp(e_1)$. Since $Sp(e_1)\subset\{z\in\mathbb{C}:|z|\leq 1\}$, we are motivated to describe the set $\hat{e}_1(S(G,K))$ within $\{z\in\mathbb{C}:|z|\leq 1\}$ in more detail.

7.13 Arnaud-Dunau polynomials. They are defined for $q\geq 1$ as polynomials $Q_n(z,q)$ in z given by

$$\begin{cases} Q_o(z|q) := 1 \\ Q_1(z|q) := z \\ nQ_n(z|q) = q(q+1)^{-1}Q_{n+1}(z|q)+(q+1)^{-1}Q_{n-1}(z|q)\ (n\geq 2). \end{cases}$$

Clearly, every $s\in S(G,K)$ admits a representation

$$\begin{cases} s[n] = Q_n(s[1]|q) \text{ for all } n\geq 1 \text{ and} \\ s[o] = 1. \end{cases}$$

So we search for all $z\in\mathbb{C}$ such that $z = s[1] = \chi_s(e_1)$ defines a homomorphism χ_s of $L^1(\mathbb{Z}_+,\tau)$ and consequently a spherical function s on (G,K).

7.13.1 For any $q\geq 1$ we define

$$\rho_q := 2q^{\frac{1}{2}}(q+1)^{-1}$$

as well as

$$\begin{cases} E_q := \{z=x+iy\in\mathbb{C} : (q-1)^2x^2+(q+1)^2y^2 \leqq (q-1)^2\} \quad (q\geqq 2) \\ E_1 := [-1,1] . \end{cases}$$

<u>Theorem.</u> The following statements are equivalent:

(i) $\sup \{|Q_n(z|q)| : n\in\mathbb{Z}_+\} \leqq 1$.

(ii) $z\in E_q$.

7.13.2 The mapping $s \to z := \chi_s(e_1)$ is a homeomorphism from $S(G,K)$ onto E_q.

From this follows that the Fourier transform $\hat{f}$ of $f\in L^1(\mathbb{Z}_+,\tau)$ can be considered as a function on E_q such that

$$\hat{f}(z) = \sum_{n\geqq o} f[n] Q_n(z|q)\tau(\{n\})$$

for all $z\in E_q$. Clearly, for $f,g\in L^1(\mathbb{Z}_+,\tau)$

$$\widehat{f*g}(z) = \hat{f}(z)\hat{g}(z)$$

whenever $z\in E_q$.

7.14 <u>Plancherel measure.</u> A spherical function $s\in S(G,K)$ is said to be *positive definite* if for all $m\geqq 1$, $t_1,\ldots,t_m\in S$ and $c_1,\ldots,c_m\in\mathbb{C}$

$$\sum_{i,j=1}^{m} c_i\overline{c_j}\ s\ [d(t_i,t_j)] \geqq o .$$

The set $Z(G,K)$ of positive definite spherical functions on (G,K) is a compact subspace of $S(G,K)$.

7.14.1 <u>Theorem:</u> For $q\geqq 2$ the following statements are equivalent:

(i) $n \to Q_n(z|q)$ belongs to $Z(G,K)$.

(ii) $z\in [-1,1]$.

Sem. on Probab.
Vol.56 1988年
P1-120

7.14.2 Theorem (Plancherel, Godement). On $Z(G,K) \cong [-1,1]$ there exists a unique measure $\pi \geq o$ such that for all $f,g \in L^1(\mathbb{Z}_+,\tau) \cap L^2(\mathbb{Z}_+,\tau)$ we have

$$\sum_{n \geq o} f[n]\overline{g[n]}\tau(\{n\}) = \int_{Z(G,K)} \hat{f}(s)\overline{\hat{g}(s)}\pi(ds).$$

The measure π is called the *Plancherel measure* of (G,K).

7.14.3 Theorem. For $q \geq 2$ the Plancherel measure π satisfies the orthogonality relation

$$\int_{[-1,1]} Q_n(z|q)Q_m(z|q)\pi(dz) = \tau(\{n\})^{-1}\delta_{m,n}$$

valid for all $n,m \geq o$, and it is explicitly given by

$$\pi(dz) = (q+1)(2\pi)^{-1}(\rho_q^2 - z^2)^{\frac{1}{2}}(1-z^2)^{-1}\,1_{[-\rho_q,\rho_q]}(z)dz.$$

7.14.4 The Fourier transformation $f \to \hat{f}$ can be extended to an isometry from $L^2(\mathbb{Z}_+,\tau)$ onto $L^2(Z(G,K),\pi)$, and for every $f \in L^1(\mathbb{Z}_+,\tau)$ we have the inversion formula

$$f[n] = \int_{[-\rho_q,\rho_q]} \hat{f}(z)Q_n(z|q)\pi(dz)$$

whenever $n \in \mathbb{Z}_+$.

For the remaining part of the section we assume that (S,A) is a homogeneous tree of degree $q \geq 2$ and (G,K) the Gelfand pair corresponding to (S,A) as described in 7.6.2.

7.15 Let $p \in L^1(\mathbb{Z}_+,\tau)$ such that $p[n] \geq o$ for all $n \in \mathbb{Z}_+$ and

Sem. on Probab.
Vol.56 1988年
P1-120

$$||p|| = \sum_{n\geq o} p[n]\tau(\{n\}) = 1.$$

The matrix $P := (p[d(s,t])_{s,t\in S}$ is symmetric, since the metric d on S is symmetric, and bistochastic, since for all $s\in S$

$$\sum_{t\in S} p[d(s,t)] = \sum_{n\geq o} p[n] \operatorname{card}(\{t\in S: d(s,t) = n\})$$

$$= \sum_{n\geq o} p[n]\tau(\{n\}) = 1.$$

Definition. The Markov chain $(X_n)_{n\in\mathbb{Z}_+}$ on the probability space $(S^{\mathbb{Z}_+}, \mathfrak{A}^{\mathbb{Z}_+}, \mathbb{P})$ with $\mathfrak{A} := \mathfrak{P}(S)$, having transition matrix P and an initial distribution μ on the state space $(S,\mathfrak{A})$ is called an *isotropic random walk associated with* p.

7.16 Remark. We note that in § 6 we treated the special case of $p\in L^1(\mathbb{Z}_+,\tau)$ with

$$p[n] := \delta_{1n}(q+1)^{-1}$$

for all $n\geq o$. The corresponding random walk was shown to be transient.

7.17 For an isotropic random walk $(X_n)_{n\in\mathbb{Z}_+}$ associated with $p \in L^1(\mathbb{Z}_+,\tau)$ the matrices $P^n = (p^{(n)}[d(s,t)])_{s,t\in S}$ of n-step transition probabilities

$$p^{(n)}[d(s,t)] = \underbrace{p*\ldots*p}_{n\text{-times}}[d(s,t)] \; (n\geq 2)$$

can be described in terms of Fourier transforms: For $n\geq 2$ and all $s,t\in S$

Sem. on Probab.
Vol.56 1988年
P1-120

$$p^{*n}[d(s,t)] = \int_{[-\rho_q,\rho_q]} (\hat{p}(z))^n Q_{d(s,t)}(z|q)\pi(dz).$$

This representation follows directly from 7.14.4 together with 7.11.3.

7.18 Theorem. Let (S,A) be a homogeneous tree of degree $q \geq 2$ with corresponding Gelfand pair (G,K) and $p \in L^1(\mathbf{Z}_+,\tau)$ with $||p|| = 1$, $p[n] \geq o$ for all $n \geq o$ and $p[o] < 1$.

Then any isotropic random walk $(X_n)_{n \in \mathbf{Z}_+}$ associated with p is transient.

Proof. We recall that a state $s \in S$ is transient iff

$$\sum_{n \geq 1} p^{*n}[d(s,s)] = \sum_{n \geq 1} p^{*n}[o] < \infty.$$

Let s be any state in S. From 7.17 we infer that for all $n \geq 1$

$$p^{*n}[d(s,s)] = p^{*n}[o]$$

$$= \int_{[-\rho_q,\rho_q]} (\hat{p}(z))^n Q_o(z|q)\pi(dz)$$

$$= \int_{[-\rho_q,\rho_q]} (\hat{p}(z))^n \pi(dz).$$

Thus it remains to be shown that

$$(*) \qquad \sum_{n \geq 1} p^{*n}[o] = \sum_{n \geq 1} \int_{[-\rho_q,\rho_q]} (\hat{p}(z))^n \pi(dz) < \infty.$$

But now

Sem. on Probab.
Vol.56 1988年
P1-120

$$||\hat{p}||_{[-\rho_q,\rho_q]} := \sup\{|\hat{p}(z)| : z \in [-\rho_q,\rho_q]\}$$

$$= \sup\{|\sum_{n\geq o} p[n]Q_n(z|q)\tau(\{n\})| : z \in [-\rho_q,\rho_q]\}$$

$$\leq \sup\{\sum_{n\geq o} p[n]\tau(\{n\})|Q_n(z|q)| : z \in [-\rho_q,\rho_q]\}$$

$$\leq \sum_{n\geq 1} p[n]\tau(\{n\})\leq 1$$

by Theorem 7.13.1. Next we show that

$$|Q_n(z|q)|<1$$

for all $n\geq 1$ and all $z \in [-\rho_q,\rho_q] \subset \overset{o}{E}_q$ which implies that $||\hat{p}||<1$.

In fact, let $z_o \in \overset{o}{E}_q$ such that $Q_n(z_o|q)=1$ for some $n\geq o$. Then the maximum principle applied to the analytic function $z \to Q_n(z|q)$ yields $Q_n(z|q)=1$ for all $z\in E_q$ and hence $n=o$.

Therefore the series $\sum_{n\geq 1} (\hat{p}(z))^n$ converges on $[-\rho_q,\rho_q]$ absolutely and uniformly towards the function F defined by

$$F(z) := \hat{p}(z)(1-\hat{p}(z))^{-1}$$

for all $z\in [-\rho_q,\rho_q]$. Thus

$$\sum_{n\geq 1} \int_{[-\rho_q,\rho_q]} (\hat{p}(z))^n \pi(dz) = \int_{[-\rho_q,\rho_q]} F(z)\pi(dz)<\infty,$$

and (*) has been shown. $\_\_|$

Sem. on Probab.
Vol.56 1988年
P1-120

§ 8 Sawyer's transience theorem

Let (S,A) be a homogeneous tree of degree $q \geqq 2$, and let (G,K) denote the corresponding Gelfand pair. We have shown the identifications $G/K \cong S$, and $G//K \cong \mathbb{Z}_+$. The Haar measure of G induces a measure τ on $\mathbb{Z}_+$ which is given by

$$\begin{cases} \tau(\{n\}) = (q+1)q^{n-1} & \text{for all } n \geqq 1 \quad \text{and} \\ \tau(\{o\}) = 1. \end{cases}$$

The subsequent result includes Theorem 7.18.

8.1 Theorem (S. Sawyer). Let $p \in L^1(\mathbb{Z}_+,\pi)$ such that $p[n] \geqq o$ for all $n \in \mathbb{Z}_+$, $||p|| = 1$, and $p[o] < 1$, and let $(X_n)_{n \in \mathbb{Z}_+}$ denote the isotropic random walk corresponding to p with initial distribution ε_o (where $o \in S$).

Then for every $s \in S$ we have

$$\lim_{n\to\infty} \frac{d(s,X_n)}{n} = \beta \quad [\mathbb{P}] ,$$

where

$$\beta := \sum_{n \geqq o} [n-2q \ (q^2-1)^{-1}(1-q^{-n})]\tau(\{n\})p[n].$$

Since $\beta > o$, we have in particular that $(X_n)_{n \in \mathbb{Z}_+}$ is transient.

Our first aim is to establish a formula which enables us to calculate horocycles for ends $b \in B$.

8.2 Lemma. Let u be a given edge. Then there exists a unique integer $m[u]$ such that

Sem. on Probab.
Vol.56 1988年
P1-120

$$d(s_{m[u]},u)<d(s_n,u)$$

for all $n\neq m[u]$, $n\in\mathbb{Z}$.

<u>Proof.</u> We make the assumption that there exist $m,n\in\mathbb{Z}$, $m\neq n$ such that

$$d(s_m,u) = d(s_n,u).$$

Then $c(s_m,s_n).c(s_n,u)$ is a path connecting s_m with u. By Theorem 1.6 we obtain

$$\begin{aligned} d(s_m,u) &= \ell(c(s_m,s_n).c(s_n,u)) \\ &= d(s_m,s_n) + d(s_n,u) \\ &= d(s_m,s_n) + d(s_m,u) \end{aligned}$$

and therefore $d(s_m,s_n) = o$ which is a contradiction to $s_m\neq s_n$.

Now we consider the set

$$I := \{m\in\mathbb{Z} : d(s_m,u)\leqq d(s,u)\}.$$

From the preceding discussion we conclude that I is finite, i.e. there exists

$$z := \min_{m\in I} d(s_m,u),$$

and again by the above there exists only one $m\in\mathbb{Z}$ satisfying

$$d(s_m,u) = z.$$

Putting $m[u] := m$ we obtain the assertion. __|

8.3 <u>Lemma.</u> For any $t_1,t_2\in S$ we have

$$\begin{aligned} \delta_b(t_2,t_1) = d(t_2,s_{m[t_2]}) + m[t_2] \\ - d(t_1,s_{m[t_1]}) - m[t_1]. \end{aligned}$$

Proof. We consider three cases:

1. Let $m[t_1] \geqq o$ and $m\ [t_2] \geqq o$.

Then the arcs $w(t_1,b)$ and $w(t_2,b)$ intersect s, and

$$d(t_i,s) = d(t_i,s_{m[t_i]}) + m[t_i]$$

for i=1,2. Thus

$$\delta_b(t_2,t_1) = d(t_2,s) - d(t_1,s)$$

$$= d(t_2,s_{m[t_2]}) + m[t_2] - d(t_1,s_{m[t_1]}) - m[t_1].$$

2. Let $m\ [t_1] \leqq o$ and $m[t_2] \leqq o$.

Then the arcs $w(t_1,b)$ and $w(t_2,b)$ intersect s_m, where

$$m := \min\{m[t_1],\ m[t_2]\}.$$

Without loss of generality let $m=m[t_1]$. Then

$$\delta_b(t_2,t_1) = d(t_2,s_{m[t_1]}) - d(t_1,s_{m[t_1]})$$

$$= d(t_2,s_{m[t_2]}) + m[t_2] - m[t_1] - d(t_1,s_{m[t_1]}).$$

3. Let $m[t_1] \geqq o$ and $m[t_2] \leqq o$ (or analoguously $m[t_1] \leqq o$ and $m[t_2] \geqq o$).

In this case the arcs $w(t_1,b)$ and $w(t_2,b)$ intersect $s_{m[t_2]}$, and

$$\delta_b(t_2,t_1) = d(t_2,s_{m[t_2]}) - d(t_1,s_{m[t_2]})$$

$$= d(t_2,s_{m[t_2]}) - d(t_1,s_{m[t_1]}) - m[t_1] + m[t_2]. \quad \square$$

In order to prove the theorem we need to establish an appropriate decomposition of the random variable

$$T_n := \frac{1}{n} d(s,X_n)$$

Sem. on Probab.
Vol.56 1988年
P1-120

for $n \geq 1$. Such a decomposition will be provided by the following

8.4 Proposition. Let $(X_n)_{n \in \mathbb{Z}_+}$ be the isotropic random walk given in the theorem, and let $s \in S$.

Then there exist a sequence $(Y_n)_{n \in \mathbb{Z}_+}$ of independent identically distributed integer-valued random variables and a sequence $(Z_n)_{n \in \mathbb{Z}_+}$ of even-integer-valued random variables on $(S^{\mathbb{Z}_+}, \mathfrak{A}^{\mathbb{Z}_+}, \mathbb{P})$ satisfying the following conditions

(i) $d(X_n, s) = d(o, s) + \sum_{k=o}^{n-1} (Y_k + Z_K)$ $\mathbb{P}$-a.s.

(ii) $E[\min(Y_k, o)] > -\infty$

(iii) $E(Y_k) = \beta$

(iv) $o \leq Z_n \leq 2d(X_n, X_{n+1})$

(v) $\mathbb{P}[Z_n = 2k \mid d(X_n, W) = d] \leq 2q^{-(d+k)}$

for all $k \in \mathbb{N}$, where

$W := \{s_m : m \leq o\}$ (with $s_o = s$)

and

$d(t, W) := \min_{m \leq o} d(t, s_m)$

for all $t \in S$.

Proof. We are given an end $b \in B$, and we define for every $n \in \mathbb{Z}_+$ a random variable

$Y_n := \delta_b(X_{n+1}, X_n)$.

Sem. on Probab.
Vol.56 1988年
P1-120

(1) It will be shown that the random variables Y_n, $n\in\mathbb{Z}_+$, satisfy the conditions (ii) and (iii) of the proposition.

Since $(X_n)_{n\in\mathbb{Z}_+}$ is a homogeneous Markov chain, we obtain for every $n\in\mathbb{Z}_+$ that

$$\mathbb{P}[d(X_n, X_{n+1}) = k] = \tau(\{k\})p[k]$$

whenever $k\in\mathbb{N}$.

(1.1) We have to compute

$$\mathbb{P}[Y_n = \ell \mid d(X_n, X_{n+1}) = k]$$

for all $k\in\mathbb{N}$ and $\ell\in\mathbb{Z}$. Under the assumption that $d(X_n, X_{n+1}) = k$ the random variable X_{n+1} attains one of the $\tau(\{k\}) = q^{k-1}(q+1)$ vertices at a distance k from X_n. The choice of one of these vertices can be considered as the end point of a path of length k whose m-th vertex has distance m from X_n $(o\leqq m\leqq k)$ (See Theorem 1.6). Since the vertices have to be pairwise different, we have q+1 possibilities for the first step and q possibilities for each of the remaining k-1 steps.

In the first step we get

$$\delta_b(z, X_n)\in\{-1, 1\}$$

and

$$\mathbb{P}\;[\delta_b(z, X_n) = 1] = q(q+1)^{-1},$$

but

$$\mathbb{P}\;[\delta_b(z, X_n) = -1] = (q+1)^{-1}$$

[since $\delta_b(z, X_n) = 1$ for q of the q+1 neighboring vertices z of X_n]. This means that the probability of growth of $\delta_b(z, X_n)$ equals $q(q+1)^{-1}$.

Sem. on Probab.
Vol.56 1988年
P1-120

Moreover, the probability of growth of $\delta_b(z,X_n)$ in any of the remaining steps equals $(q-1)q^{-1}$
[since only q choices remain open].

Here growth means increase and its opposite decrease by 1 (See 3.7.2).

Finally we have that if $\delta_b(z,X_n)$ increases in the m-th step, then there is also increase in the (m+1)st step ($1 \leqq m \leqq k-1$)
[since otherwise we would have a contradiction to the distinction of the vertices].

Thus

$$\mathbb{P}[Y_n = k \mid d(X_n,X_{n+1}) = k] = q(q+1)^{-1},$$

i.e. we have growth in all k steps. For $1 \leqq m \leqq k-1$

$$\mathbb{P}[Y_n = k-2m \mid d(X_n,X_{n+1}) = k] = (q+1)^{-1}q^{1-m}(q-1)q^{-1}$$

[For $Y_n = k-2m$ there is decrease in the first m steps and increase in the remaining k-m steps.]

Finally

$$\mathbb{P}[Y_n = -2k \mid d(X_n,X_{n+1}) = k] = (q+1)^{-1}q^{1-k},$$

i.e. we have growth in none of the k steps.

Note that under the assumption $d(X_n,X_{n+1}) = k$ one has

$$|Y_n| \leqq k \qquad \mathbb{P}\text{-a.s.}.$$

(1.2) We wish to determine

$$E(Y_n \mid d(X_n,X_{n+1}) = k)$$

Sem. on Probab.
Vol.56 1988年
P1-120

for all $k \in \mathbb{N}$. Applying the formulae of (1.1) we conclude that

$$E(Y_n \mid d(X_n, X_{n+1}) = k)$$

$$= \sum_{m=o}^{k} (k-2m)\mathbb{P}\left[Y_n = k-2m \mid d(X_n, X_{n+1}) = k\right]$$

$$= k\frac{q}{q+1} + \sum_{m=1}^{k-1} \frac{1}{q+1}\left(\frac{1}{q}\right)^{m-1}\left(\frac{q-1}{q}\right)(k-2m) - k\frac{1}{q+1}\left(\frac{1}{q}\right)^{k-1}$$

$$= k\left[\frac{q}{q+1} + \sum_{m=1}^{k-1} \frac{1}{q+1}\left(\frac{1}{q}\right)^{m-1}\left(\frac{q-1}{q}\right) + \frac{1}{q+1}\left(\frac{1}{q}\right)^{k-1}\right]$$

$$- 2\sum_{m=1}^{k-1} m\,\frac{1}{q+1}\left(\frac{1}{q}\right)^{m-1}\left(\frac{q-1}{q}\right) - 2k\frac{1}{q+1}\left(\frac{1}{q}\right)^{k-1}$$

$$= k - 2\sum_{m=1}^{k} m\,\frac{1}{q+1}\left(\frac{1}{q}\right)^{m-1}\left(\frac{q-1}{q}\right)$$

$$- 2k\,\frac{1}{q+1}\left(\frac{1}{q}\right)^{k-1}\left[1 - \frac{q-1}{q}\right]$$

$$= k - 2k\left[\frac{1}{q+1}\left(\frac{1}{q}\right)^{k}\right] - 2\,\frac{1}{q+1}\,\frac{q-1}{q}\sum_{m=1}^{k} m\left(\frac{1}{q}\right)^{m-1}$$

$$= k - 2k\left[\frac{1}{q+1}\left(\frac{1}{q}\right)^{k}\right]$$

$$- 2\,\frac{1}{q+1}\,\frac{q-1}{q}\left[\frac{1-k\left(\frac{1}{q}\right)^{k}}{1-\frac{1}{q}} + \frac{\frac{1}{q}\left(1-\left(\frac{1}{q}\right)^{k-1}\right)}{\left(1-\frac{1}{q}\right)^{2}}\right]$$

$$= k - \frac{2}{q+1} - 2\,\frac{1}{q+1}\left[\frac{q^{k-1}-1}{(q-1)q^{k-1}}\right]$$

$$= k - 2\left[\frac{1}{q+1}\left(1 + \frac{1}{q-1} - \frac{1}{(q-1)q^{k-1}}\right)\right]$$

$$= k - 2q\left[\frac{1-q^{-k}}{q^{2}-1}\right]$$

(1.3) $\mathbb{P}\left[Y_n=o \mid d(X_n;X_{n+1})=k\right]$

$$= \sum_{m=\left[\frac{k+1}{2}\right]}^{k} \mathbb{P}\left[Y_n=m \mid d(x_n,X_{n+1})=k\right]$$

$$= \sum_{m=\left[\frac{k+1}{2}\right]}^{k-1} \frac{1}{q+1}\left(\frac{1}{q}\right)^{m-1}\left(\frac{q-1}{q}\right) + \frac{1}{q+1}\left(\frac{1}{q}\right)^{k-1} \quad \text{(by 1.2)}$$

$$= \frac{1}{q+1}\left(\frac{q-1}{q}\right)\left[\sum_{m=o}^{k-2}\left(\frac{1}{q}\right)^m - \sum_{m=o}^{\left[\frac{k+1}{2}\right]-1}\left(\frac{1}{q}\right)^m\right] + \frac{1}{q+1}\left(\frac{1}{q}\right)^{k-1}$$

$$= \frac{1}{q+1}\left(\frac{1}{q}\right)^{\left[\frac{k+1}{2}\right]}$$

$$\leqq \frac{1}{q+1}\left(\frac{1}{q}\right)^{\frac{k}{2}} .$$

(1.4) Now we determine the distribution of Y_n for $n \in \mathbb{Z}_+$. Since Y_n is discrete-valued, it suffices to determine $\mathbb{P}_{Y_n}(\{m\})$ for every $m \in \mathbb{Z}$. From the formula in Lemma 8.4. we conclude

that

$$\mathbb{P}\left[Y_n = m \mid d(X_n,X_{n+1}) = o\right] = \delta_{om}$$

and hence that

$$\mathbb{P}\left[Y_n=m\right] = \sum_{k \geqq o} \mathbb{P}\left[d(X_n,X_{n+1}) = k\right] \cdot \mathbb{P}\left[Y_n = m \mid d(X_n,X_{n+1}) = k\right]$$

$$= \sum_{k \geqq o} \tau(\{k\})\; p\left[k\right] c(k,m)$$

[by the formula preceding (1.1)],

Sem. on Probab.
Vol.56 1988年
P1-120

where c(k,m) by (1.1) depends only on k and m, and not on n. Thus the distribution of Y_n does not depend on n, i.e. the sequence $(Y_n)_{n\in\mathbb{Z}_+}$ is identically distributed. Moreover $(Y_n)_{n\in\mathbb{Z}_+}$ is an independent sequence.

(1.5) We shall show (ii) of the theorem.

Clearly, min $(Y_k, o) = -Y_k^- = -\max(-Y_k, o)$. The assertion is equivalent to the integrability of Y_k^- for all $k\geq o$, and the latter follows from the following chain of inequalities:

$$E\left[\max(-Y_k, o)\right]$$

$$= \sum_{k\geq o} k\, \mathbb{P}_{-Y_n}(\{k\})$$

$$= \sum_{k\geq o} k\, \mathbb{P}\left[Y_n = -k\right]$$

$$= \sum_{k\geq o} k \sum_{\ell\geq o} \tau(\{\ell\})\, p[\ell]\, \mathbb{P}\left[Y_n = -k \mid d(X_n, X_{n+1}) = \ell\right]$$

(by 1.3)

$$\leq \sum_{\ell\geq o} \ell . c\left(\frac{1}{q}\right)^{\frac{k}{2}} \quad \text{with a constant } c$$

(by 1.3 and $|Y_n| \leq \ell$ whenever $d(X_n, X_{n+1}) \leq \ell$)

$$= c \sum_{\ell\geq o} \ell\left(\frac{1}{\sqrt{q}}\right)^{\ell}$$

$$\leq c \sum_{\ell\geq o} (\ell+1)\left(\frac{1}{\sqrt{q}}\right)^{\ell}$$

$$= \frac{cq}{(\sqrt{q}-1)^2} < \infty.$$

Sem. on Probab.
Vol.56 1988年
P1-120

(1.6) Now we turn to the proof of (iii) of the theorem.

$$E(Y_n) = \sum_{k\geq o} \mathbb{P}[d(X_n,X_{n+1})=k]\, E[Y_n \,|\, d(X_n,X_{n+1})=k]$$

$$= \sum_{k\geq o} \tau(\{k\})\;\; p[k]\,[k-2q(1-q^{-k})(q^2-1)^{-1}]$$

(by the formula preceding (1.1) and (1.2) respectively)

$$= \beta$$

So far we established the properties of the sequence $(Y_n)_{n\in Z_+}$.

(2) It will be shown that the sequence $(Z_n)_{n\in Z_+}$ properly defined satisfies conditions (i), (iv) and (v).

We define Z_n so that (i) holds, i.e. for each $n\geq o$

$$Z_n := d(X_{n+1},s) - d(X_n,s) - Y_n.$$

Indeed, with this definition we get

$$\sum_{k=o} (Y_k+Z_k)$$

$$= \quad Y_o - d(X_o,s) + d(X_1,s) \qquad\qquad -Y_o$$

$$+\, Y_1 \qquad -d(X_1,s)+d(X_2,s) \qquad\qquad -Y_1$$

$$+\, Y_2 \qquad\qquad -d(X_2,s)+d(X_3,s) \qquad -Y_2$$

$$\vdots$$

$$+\, Y_{n-1} \qquad\qquad\qquad -d(X_{n-1},s)+d(X_n,s) \qquad -Y_{n-1}$$

$$= d(X_n,s)-d(X_o,s)$$

$$= d(X_n,s)-d(o,s) \quad \mathbb{P}\text{-a.s.}.$$

Sem. on Probab.
Vol.56 1988年
P1-120

(2.1) The following statements are equivalent:

(α) $Z_n \neq o$

(β) The geodesic $c(X_n, X_{n+1})$ from X_n to X_{n+1} and the infinite chain $(s_{-n})_{n \geq o}$ have at least one edge in common.

First suppose that $c(X_n, X_{n+1})$ and $(s_{-n})_{n \geq o}$ have no common edge. Then

$$m[X_n] = m[X_{n+1}]$$

and hence

$$Z_n = d(X_{n+1}, s) - d(X_n, s) - \delta_b(X_{n+1}, X_n)$$

$$= d(X_{n+1}, s) - d(X_n, s)$$

$$-d(X_{n+1}, s_{m[X_{n+1}]}) - m[X_{n+1}]$$

$$+d(X_n, s_{m[X_n]}) + m[X_n]$$

(by Lemma 8.3)

$$= d(X_{n+1}, s_{m[X_n]}) + d(s_{m[X_n]}, s_o)$$

$$- d(X_n, s_{m[X_n]}) - d(s_{m[X_n]}, s_o)$$

$$- d(X_{n+1}, s_{m[X_n]}) + d(X_n, s_{m[X_n]})$$

$$= o$$

which contradicts the hypothesis.

Conversely we suppose that (β) holds. Without loss of generality we assume that

$$m[X_{n+1}] \leqq m[X_n].$$

Let $[s_m, \ldots, s_{m+k}]$ denote the edge common to $c(X_n, X_{n+1})$ and $(s_n)_{n \leq o}$, where $m \leqq -1$, $k \geqq 1$.

[The totality of these edges is a finite interval in the sense of 1.12.]

Then

$$S_m[X_{n+1}] = s_m, \quad s_m[X_n] = s_{m+k}$$

and

$$d(X_{n+1}, s) - d(X_n, s)$$

$$= d(X_{n+1}, s_m) + |m[X_{n+1}]| - d(X_n, s_{m+k}) - |m[X_n]|.$$

Moreover,

$$Y_n = \delta_b(X_{n+1}, X_n)$$

$$= d(X_{n+1}, s_m) - d(X_n, s_m)$$

$$= d(X_{n+1}, s_m) - d(X_n, s_{m+k}) - k,$$

hence

$$Z_n = d(X_{n+1}, s_m) + |m[X_{n+1}]| - d(X_n, s_{m+k})$$

$$-|m[X_n]| - d(X_{n+1}, s_m) + d(X_n, s_{m+k}) + k$$

$$= |m[X_{n+1}]| - |m[X_n]| + k .$$

1st case: $m+k \leqq o$

Then $Z_n = -m+m+k+k = 2k \neq o$.

2nd case: $m+k > o$. Putting $m^* := -m$ we get

Sem. on Probab.
Vol.56 1988年
P1-120

$$Z_n = -m-m-k+k = -2m = 2m^*>o.$$

Thus we obtain (i).

(2.2) Property (iv) results directly from the second part of the proof of (2.1), since Z_n is always positive and even (integer) and

$$o \leqq Z_n \leqq 2k \leqq 2d(X_n, X_{n+1}).$$

More precisely we obtain that the value of Z_n equals twice the number of common edges of $c(X_n, X_{n+1})$ and $(s_n)_{n \leqq o}$.

(2.3) Property (v) follows from the subsequent inequalities valid for any $k>o$ (by applying (2.1)):

$$\mathbb{P}\left[Z_n = 2k \mid d(X_n, W) = d\right] = \left(\frac{1}{q+1}\right)^d \left(\frac{2}{q+1}\right)^k \leqq 2q^{-(d+k)}.$$

The proposition has been proved. $\_\_|$

In order to complete the proof of Theorem 8.1 we need another

8.5 <u>Proposition</u>. Under the hypotheses of Theorem 8.1 we have

$$\sum_{n \geqq o} Z_n < \infty \quad \mathbb{P}\text{-a.s.}.$$

<u>Proof.</u> 1. First we show that for any $t \in S$

$$\sum_{n \geqq 1} p^{(n)}\left[d(o,t)\right] \leqq Cq^{-\frac{d(s,t)}{2}},$$

where C denotes a positive constant.

In fact we note that

$$\sum_{n \geqq 1} p^{(n)}\left[d(o,t)\right] < \infty$$

which follows in analogy to the previous transience theorem 7.18, where we applied the formula 7.17 for the n-step transition probabilities. By this very formula we also get that

$$p^{(n)}[d(o,t)] = p^{*n}[d(o,t)]$$

$$= \int_{-\rho_q}^{\rho_q} \hat{p}(z)^n Q_{d(o,t)}(z|q)\sigma(dz)$$

$$= \int_0^{\pi} \hat{p}(\rho_q\cos\Theta)^n Q_{d(o,t)}(\rho_q\cos\Theta|q)$$

$$\rho_q{}^3\sin^3\Theta\ \frac{q+1}{2\pi}(1-\rho_q{}^2\cos^2\Theta)^{-1}d\Theta$$

[by the substitution $z = \rho_q \cos\Theta$].

But in calculating the explicit form of the Plancherel measure π (Theorem 7.14.3) we see that

$$Q_{d(o,t)}(\rho_q\cos\Theta|q) = q(q+1)^{-1}q^{-\frac{d(o,t)}{2}}(e^{in\Theta}+e^{-in\Theta})$$

and hence that

$$p^{(n)}[d(o,t)] = q^{-\frac{d(o,t)}{2}}\int_0^{\pi}\hat{p}(\rho_q\cos\Theta)^n$$

$$\cdot\ \frac{q}{2\pi}(e^{in\Theta}+e^{-in\Theta})\rho_q{}^3\sin^3\Theta(1-\rho_q{}^2\cos^2\Theta)d\Theta.$$

Altogether this implies

$$\infty > \sum_{n\geq 1} p^{(n)}[d(o,t)] = q^{-\frac{d(o,t)}{2}}\tilde{C}$$

with

$$\tilde{C} := \sum_{n\geq 1}\int_0^{\pi}\hat{p}(\rho_q\cos\Theta)^n\ \frac{q}{2\pi}(e^{in\Theta}+e^{-in\Theta})\rho_q{}^3\sin^3\Theta(1-\rho_q{}^2\cos^2\Theta)d\Theta,$$

Sem. on Probab.
Vol.56 1988年
P1-120

whence

$$\tilde{C}\, q^{-\frac{d(o,t)}{2}} \leqq C\, q^{-\frac{d(s,t)}{2}}$$

with

$$C := \tilde{C}\, q^{-\frac{d(o,s)}{2}}$$

2. (Estimate for $E(Z_n)$).

$$E(Z_n) = E(E^{X_n}(Z_n))$$

$$= E(E(\sum_{k\geqq 1} 2k 1_{[Z_n=2k]} | X_n))$$

[by Property (iv) of Proposition 8.5]

$$= E(\sum_{k\geqq 1} 2kE(1_{[Z_n=2k]} | X_n))$$

$$= \sum_{k\geqq 1} 2kE(E(1_{[Z_n=2k]} | X_n))$$

$$= \sum_{k\geqq 1} 2kE(E^{d(X_n,W)}(E^{X_n}(1_{[Z_n=2k]})))$$

$$= \sum_{k\geqq 1} 2kE(E^{d(X_n,W)}(1_{[Z_n=2k]}))$$

[since $\mathfrak{A}(d(X_n,W)) \subset \mathfrak{A}(X_n)$]

$$= E(\sum_{k\geqq 1} 2k\, \mathbb{P}[Z_n=2k | d(X_n,W)])$$

$$\leqq 2\; E(\sum_{k\geqq 1} 2k\, q^{-d(X_n,W)} q^{-k})$$

[by Property (v) of Proposition 8.5]

$$= 2\; E(2q^{-d(X_n,W)} \sum_{k\geqq 1} k\, q^{-k})$$

$$\leqq C_1\; E(q^{-d(X_n,W)}),$$

where C_1 denotes a constant independent of n.

3. (Estimate for $E(\sum_{n\geq o} Z_n)$).

$$E(\sum_{n\geq o} Z_n) \leqq C_1 \sum_{n\geq o} E(q^{-d(X_n,W))}$$

$$= C_1 \sum_{n\geq o} \sum_{t\in S} q^{-d(t,w)} p^{(n)}[d(o,t)]$$

$$= C_1 q^{-d(o,W)} + C_1 \sum_{t\in S} q^{-d(t,W)} (\sum_{n\geq 1} p^{(n)}[d(o,t)])$$

$$\leqq C_1 p^{-d(o,W)}$$

$$+ C_2 \sum_{t\in S} q^{-d(t,W)} q^{-\frac{d(s,t)}{2}}$$

[by Part 1. of the proof]

$$\leqq C_1 + C_2 \sum_{d\geq o} q^{-\frac{d}{2}} \sum_{\{t\in S: d(s,t)=d\}} q^{-d(t,W)}$$

with an additional constant $C_2 > o$.

Since $s\in S$ has been held fixed, we have

$$d(s,t) = d \text{ and } d(t,W) = m$$

iff $s_{m[t]} = s_{m-d}$

[By definition of $d(t,W)$ and W, $d(t,W) \leqslant d(s,t)$ for all $t\in S$.]

Then

$$\sum_{\{t\in S: d(s,t)=d\}} q^{-d(t,W)} = \sum_{m=o}^{d} q^{-m} \Big(\sum_{\{t\in S: d(s,t)=d \text{ and } d(t,W)=m\}} 1 \Big)$$

$$= \sum_{m=o}^{d} q^{-m} q^{m-1} (q+1),$$

Sem. on Probab.
Vol.56 1988年
P1-120

since $|\{t\in S: d(s,t)=d \text{ and } d(t,W)=m\}|$

$$= q^{m-d-1}(q+1)$$

$$\leq q^{m-1}(q+1)$$

$[d\geq o,\ q>1]$

$$= \sum_{m=o}^{d}(1+\frac{1}{q})$$

$$\leq 2(d+1).$$

So we continue the above chain of inequalities and get

$$E(\sum_{n\geq o} Z_n) \leq C_1 + 2C_2 \sum_{d\geq o}(d+1)(\frac{1}{\sqrt{q}})^d$$

$$= C_1 + 2C_2 \frac{1}{(1-\frac{1}{\sqrt{q}})^2}$$

$$< \infty.$$

4. Since $E(\sum_{n\geq o} Z_n)<\infty$, we conclude trivially that

$$\sum_{n\geq o} Z_n < \infty \quad \mathbb{P}\text{-a.s.} \quad \_\_|$$

<u>Proof of Theorem</u> 8.1. For every $n\geq 1$ we have

$$\frac{1}{n}d(X_n,s) = \frac{1}{n}d(o,s) + \frac{1}{n}\sum_{k=o}^{n-1} Y_k + \frac{1}{n}\sum_{k=o}^{n-1} Z_k .$$

This decomposition follows from Proposition 8.4.

We shall show that

(1) $$\frac{1}{n}\sum_{k=o}^{n-1} Y_k \to \beta \qquad \mathbb{P}\text{-a.s.}$$

and

(2) $$\frac{1}{n}\sum_{k=o}^{n-1} Z_k \to o \qquad \mathbb{P}\text{-a.s.}$$

For (1) we will discuss two cases separately.

If $o<\beta<\infty$ then by (ii) of Proposition 8.4 Y_k is integrable for all $k\geq o$ and $E(Y_k)=\beta$. But $(Y_k)_{k\geq o}$ is a sequence of independent and identically distributed real-valued random variables. Thus $(Y_k)_{k\geq o}$ satisfies the hypotheses of the strong law of large numbers, whence

$$\frac{1}{n}\sum_{k=o}^{n-1} Y_k \to E(Y_1) = \beta \qquad \mathbb{P}\text{-a.s.}$$

If $\beta = +\infty$ then Y_k is quasi-integrable by (i) and (ii) of Proposition 8.4 and clearly

$$\frac{1}{n}\sum_{k=o}^{n-1} Y_k \to \infty \qquad \mathbb{P}\text{-a.s.}$$

by the converse of the strong law of large numbers.

Concerning (2) we just apply Property (iv) of Proposition 8.4 in order to get that $Z_n\geq o$ for all $n\geq o$, and Proposition 8.5 in order to obtain that

$$\sum_{n\geq o} Z_n < \infty \qquad \mathbb{P}\text{-a.s.}$$

Consequently $\quad \frac{1}{n}\sum_{k=o}^{n-1} Z_k \to o \qquad \mathbb{P}\text{-a.s.}$ $\_\_|$

Selected References

[1] Arnaud, J.-P.: Fonctions sphériques et fonctions définies positives sur l'arbre homogène. C.R. Acad.Sc. 290, Série A (1980), 99-101

[2] Cartier, P.: Géométrie et analyse sur les arbres. Séminaire Bourbaki, 24e année, exposé no 407 (Février 1972), 123-140

[3] Cartier, P.: Fonctions harmoniques sur un arbre. Istituto Nazionale di Alta Matematica, Symposia Mathematica IX (1972), 203-270

[4] Cartier, P.: Harmonic analysis on trees. Proc.Symp. in Pure Math. AMS Vol. XXVI (1973), 419-424

[5] Dieudonné, J.: Grundzüge der modernen Analysis Band 5/6. Vieweg-Verlag, Braunschweig-Wiesbaden 1979

[6] Dunau, J.-L.: Etude d'une classe de marches aléatoires sur l'arbre homogène. Publications du Laboratoire de Statistique de l'Université de Paul Sabatier, no 04-1976, Toulouse

[7] Gerl, P.: Wachstum und Irrfahrten bei Gruppen und Graphen. Publ. IRMA Strasbourg (1984), 1-38

[8] Koranyi, A., Picardello, M.A.: Boundary behaviour of eigen-functions of the Laplace operator on trees. Ann.Sc.Norm.Sup. Pisa 13 (1986), 389-319

[9] Letac, G.: Problèmes classiques de probabilité sur un couple de Gelfand. In: Analytic Methods in Probability Theory (edited by D. Dugué, E. Lukacs, V.K. Rohatgi), pp. 93-120. Proceedings, Oberwolfach, Germany (1980), Lecture Notes in Math.Vol. 861, Springer 1981

[10] Letac, G.: Dual random walks and special functions on homogeneous trees.
In: Marches Aléatoires et Processus Stochastiques Sur les Groupes de Lie, pp. 96-142
Journées S.M.F. Nancy (1981), Equipe Associée d'Analyse Globale no 839
Institut Elie Cartan 1983

[11] Letac, G.: Les fonctions sphériques d'un couple de Gelfand symétrique et les chaînes de Markov.
Adv.Appl.Prob. 14 (1982), 272-294

[12] Letac, G., Arnaud, J.-P., Dunau, J.-L.: Arbres homogènes et couples de Gelfand.
Publications du Laboratoire de Statistique et Probabilités de l'Université Paul Sabatier, Toulouse, no 02 (1983)

[13] Picardello, M.A., Woess, W.: Martin boundaries of random walks: ends of trees and groups.
Trans.Amer.Math.Soc. 302 (1987), 185-205

[14] Picardello, M.A., Woess, W.: Random walks, harmonic functions, and ends of graphs.
Arbeitsbericht 3/1986, Institut für Mathematik und Angewandte Geometrie, Montanuniversität Leoben, A-8700 Leoben, Austria, 18 pages

[15] Sawyer, S.: Isotropic random walks in a tree.
Z.Wahrscheinlichkeitstheorie verw.Gebiete 42 (1978), 279-292

[16] Sawyer, S., Steger, T.: The rate of escape for an anisotropic random walk in a tree.
Prob.Th.Rel.Fields 76 (1987), 207-230