Sem. on Probab.
Vol.36 1972年
P1-89

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Transversal Commutation Relation and its Applications to Ergodic Theory

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TRANSVERSAL COMMUTATION RELATION

and its APPLICATIONS to ERGODIC THEORY

Masasi KOWADA

Seminair on Probability Vol. 36

確率論セミナー

Sem. on Probab.
Vol.36 1972年
P1-89

Table of Contents

Sem. on Probab.
Vol.36 1972年
P1-89

Sem. on Probab.
Vol.36 1972年
P1-89

Preface

The purpose of this monograph is to show what roles the transversal commutation relations play in many branches of mathematics especially in ergodic theory. Let G_1 and G_2 be two groups and let $\{T_g\}$ and $\{Z_h\}$ be the groups of transformations on a space X such that

$$T_g Z_h = Z_{\tau_g(h)} T_g \qquad g \in G_1, \quad h \in G_2$$

where τ_g is an isomorphism on G_2 for each $g \in G_1$. We call the above commutation relation the transversal commutation relation and $\{Z_h\}$ is said to be the transversal group of $\{T_g\}$. The space X is sometimes taken as a Hilbert space, the n-dimensional eucllidean space, or as a probability space. It seems to me the terminologies of the transverse and the transversal flow were firstly introduced by Ya. G. Sinai [21] to prove that a flow of a billiards model is a K-flow, although the idea of the transverse can be found ambiguously in E.Hopf [7] ; this monograph is muchly inspired by them.

There are the three kinds of approaches to the transversal commutation relation. Firstly the object of study is very the group $\{T_g\}$, and $\{Z_h\}$ is introduced as a tool of studying $\{T_g\}$.

This view-point is taken in the chapter IV to study the spectral type of a 1-parameter group of unitary operators $\{T_g\}$ on a Hilbert space and in the chapter III to the isomorphy problem of group automorphisms on the torus.

The second one is to regard the transversal commutation relation

Sem. on Probab.
Vol.36 1972年
P1-89

as a represenation

$$G_1 \overset{\tau}{\otimes} G_2 \ni g \otimes h \rightarrow Z_h T_g$$

of a skew-product group $G_1 \overset{\tau}{\otimes} G_2$; let $G_1 = G_2 = R$, the real axis, and let G be the group of transformations; $x \rightarrow \alpha x + \beta$, $\beta \in R$ and α is a positive number. The we get $G = G_1 \overset{\tau}{\otimes} G_2$, $\tau_\alpha(\beta) = \alpha\beta$ and the problem of finding all the positive definite functions on G is reduced to finding all the unitary representations. This will be done in §2.5. Refer also to G. Machey [13] .

The last one is a contrary viewpoint to the first one; the main object is the group $\{Z_h\}$ and a family of groups $\{T_g\}$ satisfying the transversal commutation relations is introduced as the dual object of $\{Z_h\}$. From this viewpoint, we deal with a measurable flow $\{Z_h\}$ appealing to the structures of the orbit-preserving transformation group, an extension of the group $\{T_g\}$, the chapter V.

We can find a transversal commutation relation in S. Lie's theory of the ordinary differential equation in which an integral factor can be constructed by making use of a transversal group. Although this classical theory is described in an introductory text book of the differential equations, we shall mention to it in §1 .

Among the interesting topics to which I have no room to mention in this monograph, I cite Ya. G. Sinai [21] , H.A.Dye. [26], [27], W.Krieger [29], Y.Ito [30] and S.Ito [31] . It is a pleasure to acknowledge my indebtedness to several members of Probability Seminair. Finally, Miss Reiko Asahina, Miss Hisae Wakayama, Miss Akiko Fujisawa typed the original manuscript, I extend my thanks for the expert services.

Tokyo September, 1972. M. Kowada

Sem. on Probab.
Vol.36 1972年
P1-89

I. The ordinary differential equations and the transversal groups.

In this § we take up the Lie's theory of the ordinary differential equation in which it can be found out that the idea of § is originated. About this § , refer to [25].

Let T_t be a transformation from R^2 into R^2 with parameter $t \in R$ defined by

$$T_t(x,y) = (x_1, y_1) \ ; \quad \left.\begin{array}{l} x_1 = \phi(x, y, t) \\ y_1 = \psi(x, y, t) \end{array}\right\} \tag{1.1}$$

,where ϕ and ψ are real analytic functions.
We assume $\{T_t\}$ form a group ;

$$T_{t_1+t_2} = T_{t_1} T_{t_2} \quad .$$

Let

$$\xi(x,y) = \frac{\partial\phi}{\partial t}(x, y, 0)$$

$$\eta(x,y) = \frac{\partial\psi}{\partial t}(x, y, 0)$$

and then we get the infinitesimal transformation U of the group $\{T_t\}$

$$Uf(x,y) = \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} \tag{1.2}$$

, where f is analytic. Conversely if $U = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ is given, we may construct the transeformation group $\{T_t\}$ which have U as its infinitesimal transformation,

$$\left.\begin{array}{l} x_1 = e^{tU}x \\ y_1 = e^{tU}y \end{array}\right\} \tag{1.3}$$

We say a function f on R^2 is invariant under the group $\{T_t\}$ if

$f(T_t(x,y)) = f(x,y)$, $(x,y) \in R^2$.

In other word, f is invariant, if f is constant on each orbit of $\{T_t\}$.

The following theorem give a characterization of invariant function by appealing to the infinitesimal transformation U.

Theorem 1.1. A function $f(x,y)$ is invariant under the group $\{T_t\}$ if and only if $Uf = 0$.

proof. Since

$$f(T_t(x,y)) - f(x,y) = \sum_{k=1}^{\infty} \frac{t^k}{k!} U^k f ,$$

$$f(T_t(x,y)) \equiv f(x,y) \quad \text{if and only if} \quad Uf = 0 .$$

Note that the invariant function is an integral of the characteristic equation

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}$$

of the partial differential equation

$$Uf = \xi(x,y) \frac{\partial f}{\partial x} + \eta(x,y) \frac{\partial f}{\partial y} = 0$$

and orbit of each point except invariant points under the group $\{T_t\}$ is given by

$$f(x,y) = c$$

, where f is an invariant function.

Sem. on Probab.
Vol.36 1972年
P1-89

Now we shall state about a transversal commutation relation which was discussed by S. Lie (1874) concerning to his theory of differential equations.

A differential equation

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1.4}$$

is said to be invariant under the group $\{T_t\}$ if

$$P(T_t(x,y))dx + Q(T_t(x,y))dy = 0$$

implies

$$P(x,y)dx + Q(x,y)dy = 0 ,$$

in other words if there exists a function $K(x,y)$ such that

$$P(T_t(x,y)) = K(x,y)P(x,y)$$

$$Q(T_t(x,y)) = K(x,y)Q(x,y) .$$

Put $T_t = e^{tU}$, where $U = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$.

<u>Theorem 1.2.</u> If a differential equation

$$P(x,y)dx + Q(x,y)dy = 0$$

is invariant under the group $\{T_t\}$, it has

$$1/(\xi P + \eta Q)$$

as an integrating factor.

<u>proof.</u> Let $\frac{\partial H}{\partial x}/P = K(x,y)$, where $H(x,y) = c$ be a solution of the given differential equation. Then by the relation $\frac{\partial H}{\partial x} : \frac{\partial H}{\partial y} = P : Q$ it follows

$$\frac{\partial H}{\partial x} = KP , \quad \frac{\partial H}{\partial y} = KQ \tag{1.5}$$

Sem. on Probab.
Vol.36 1972年
P1-89

Since we may chose a solution H such that

$$UH = \xi\frac{\partial H}{\partial x} + \eta\frac{\partial H}{\partial y} \equiv 1 \tag{1.6}$$

it follows from (1.5) and (1.6)

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy = (Pdx + Qdy)/(\xi P + \eta Q) .$$

The following theorem gives us a condition for a differential equation to be invariant under the group appealing to its transversal group.

Theorem 1.3. A differential equation

$$P(x,y)dx + Q(x,y)dy = 0$$

is invariant under the group $\{T_t\}$ if and only if there exists $\lambda = \lambda(x,y)$ such that

$$[U, V] = \lambda(x,y)V \tag{1.7}$$

, where $V = Q\frac{\partial}{\partial x} - P\frac{\partial}{\partial y}$, $e^{tU} = T_t$ and $[U, V] = UV - VU$.

proof. Let $H = H(x,y)$ be a solution such that

$$UH = \xi\frac{\partial H}{\partial x} + \eta\frac{\partial H}{\partial y} \equiv 1 .$$

Then

$$VH = QH_x - PH_y = 0$$

and

$$0 = UVH - VUH = [U, V]H$$

$$= U(QH_x - PH_y) - V(H_x + H_y)$$

$$= (UQ - V\xi)H_x - (UP + V\eta)H_y$$

Hence it follows

Sem. on Probab.
Vol.36 1972年
P1-89

$$\frac{UQ - V\xi}{Q} = \frac{UP + V\eta}{P} \equiv \lambda(x,y)$$

and

$$[U, V] = (UQ - V\xi)\frac{\partial}{\partial x} - (UP + V\eta)\frac{\partial}{\partial y}$$

$$= \lambda(x,y)\left(Q\frac{\partial}{\partial x} - P\frac{\partial}{\partial y}\right) = \lambda(x,y)V .$$

Conversely if $VH = 0$,

$$0 = \lambda VH = [U, V]H = - VUH ,$$

so that UH is also a solution of the equation

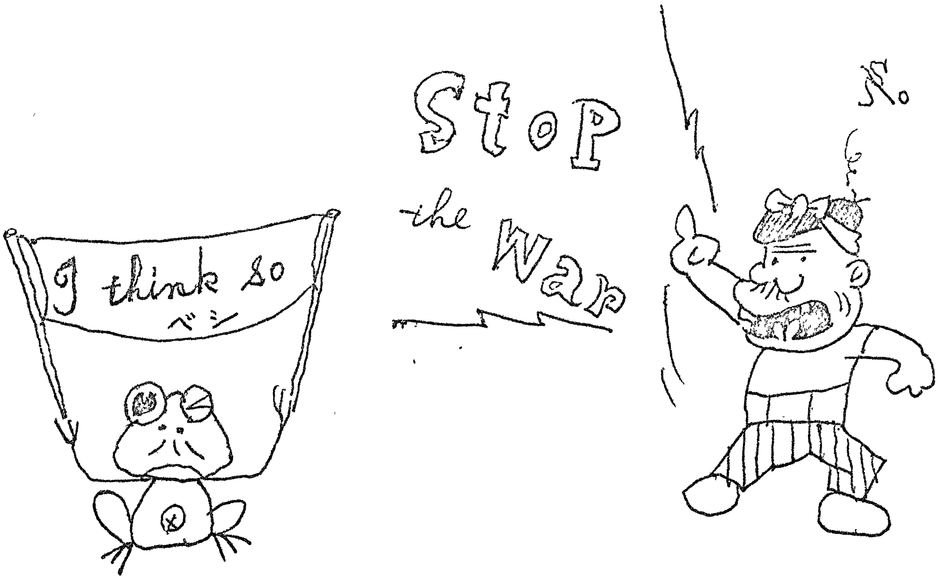
$$V_f = 0 .$$

This implies the orbit $H(x,y) = c$ is invariant under the group $\left\{T_t,\ T_t = e^{tU}\right\}$. This completes the proof. The equation (1.7) implies a transversal commutation relation

$$T_t Z_s f(x,y) = Z_{s e^{\lambda t}} T_t f(x,y) \qquad \text{a.e } (x,y)$$

where $T_t = e^{tU}$, $Z_s = e^{sV}$ and $f \in L^2$. Refer also to the example in § 4.3 .

Sem. on Probab.
Vol.36 1972年
P1-89



II Representations of the skew product group and the transversal commutation relation.

In this §, Some examples of the skew product groups which appear in many fields of mathematics are given.

The spectral analysis of the unitary representations of some of the skew product groups and their relationships to the theory of flow will be given in the other §.

§ 2.1. Skew product of groups

Definition of skew product. Let G_1 and G_2 be groups and $A(G_2)$ be the group of automorphisms of G_2. Let τ be a homomorphism of G_1 into $A(G_2)$. We define a group operation in the product $G = G_1 \times G_2$ as follows ; for $[g_1, h_1]$, $[g_2, h_2] \in G$

Sem. on Probab.
Vol.36 1972年
P1-89

$$[g_1, h_1]\,[g_2, h_2] = [g_1 g_2, h_1 \tau_{g_1}(h_2)]$$

The group $G = G_1 \overset{\tau}{\otimes} G_2$ with this operation is called a skew product group of G_1 and G_2 with respect to τ .

It is easy to see the following proposition.

Proposition 2.1.1. Suppose G_1 and G_2 are (locally compact) topological groups and introduce the product topology in the group $G = G_1 \overset{\tau}{\otimes} G_2$. Then if the mapping

$$[g, h] \to \tau_g(h)$$

is continuous, G becomes (locally compact) topological group.

§ 2.2. Representation of $G = G_1 \overset{\tau}{\otimes} G_2$

Suppose we are given a unitary representation

$$G \ni [g, h] \to \Pi_{[g,h]}$$

on a Hilbert space , Then the representations U and V

$$U : \quad G_1 \ni g \to U_g = \Pi_{[g, e_2]}$$

$$V : \quad G_2 \ni h \to V_h = \Pi_{[e_1, h]}$$

are the unitary representations on of G_1 and G_2 , respectively, where e_i is the identity of G_i $(i = 1, 2)$.

In this case $\{U_g\}$ and $\{V_h\}$ satisfy the following commutation relation;

Sem. on Probab.
Vol.36 1972年
P1-89

Proposition 2.2.1. $U_g V_h U_g^* = V_{\tau g(h)}$, $g \in G_1$, $h \in G_2$. (2.1)

proof. $\pi[g,e_2]\pi[e_1,h]\pi[g^{-1},e_2]$

$=\pi[g, e_2][e_1,h][g^{-1}, e_2]$

$=\pi[g, e_2][e_1 g^{-1}, h\tau_{e_1}(e_2)]$

$=\pi[gg^{-1}, e_{2}\tau_g(h)] = \pi[e_1,\tau_{g(h)}]$

Conversely any representations $G_1 \ni g \to U_g$ and $G_2 \ni h \to V_h$ on a Hilbert space satisfying the above commutation relation (2.1) arise from the representation of $G_1 \overset{\tau}{\otimes} G_2$ in the above way.

Proposition 2.2.2. Let there be given unitary representations of the group G_1 and G_2 on a Hilbert space $\mathcal{H}$

$$U : \quad G_1 \ni g \to U_g$$

$$V : \quad G_2 \ni h \to V_h$$

satisfying the commutation relation

$$U_g V_h U_g^* = V_{u(g,h)} \qquad u(g,h) \in G_2 .$$

If V is faithful, then there exists an isomorphism such that the representations U and V are the restrictions of the representation π of $G = G_1 \overset{\tau}{\otimes} G_2$ defined by

$$\pi[g, h] = V_n U_g .$$

proof. Put $u(g, h) = \tau_g(h)$. We get

$$U_g V_{h_1 h_2} U_g^* = U_g V_{h_1} U_g^* U_g V_{h_2} U_g^*$$

Sem. on Probab.
Vol.36 1972年
P1-89

$$= V_{\tau_g(h_1)} V_{\tau_g(h_2)}$$

$$= V_{\tau_g(h_1 h_2)}$$

and, since V is faithful, it follows

$$\tau_g(h_1 h_2) = \tau_g(h_1)\tau_g(h_2) .$$

Suppose $\tau_g(h) = e_2$, then $U_g V_h = U_g$ and hence there exists an element $f_o \in \mathfrak{H}$ for which $f_o \neq V_h f_o \equiv f_1$, unless $h = e_2$. Then

$$U_g f_o = U_g V_h f_o = U_g f_1 ,$$

this is a contradiction and hence $\tau_g \in A(G_2)$ for any $g \in G_1$.

From the equalities

$$U_{g_1 g_2} V_h U_{g_1 g_2}^* = U_{g_1} V_{\tau_{g_2}(h)} U_{g_1}^* = V_{\tau_{g_1}(\tau_{g_2}(h))}$$

$$= V_{\tau_{g_1 g_2}(h)} \quad \text{for any } h \in G_2$$

it follows $\tau_{g_1 g_2} = \tau_{g_1}\tau_{g_2}$.

<u>N. B.</u> When U and V are strongly continuous unitary representations on $\mathfrak{H}$ of topological groups G_1 and G_2 respectively and $u = u(g,h)$ is continuous mapping of $G_1 \times G_2$ into G_2, then $\pi_{[g,h]} = V_h V_g$ is a strongly continuous unitary representation of $G_1 \overset{\tau}{\otimes} G_2$. Moreover if both U and V are faithful, then π is also faithful.

§ 2.3. Examples of the skew product groups

Some of the examples given in this § will be again taken up in the

Sem. on Probab.
Vol.36 1972年
P1-89

following sections.

Example 1. Let G_1 be the group generated by a continuous group automorphism A on the n-dimensional torus T^n, and let G_2 be the torus group T^n. We define the mapping τ from G_1 into $A(G_2)$ by

$$\tau_{A^n}(h) = A^n h \ , \quad A^n \in G_1 \ , \quad h \in G_2 \ .$$

Let U, V be unitary representations of G_1 and G_2 on the Hilbert space $L^2(T^n)$ with the normalized Harr measure defined by

$$U : \quad U_{A^n} f(x) = f(A^n x)$$

$$V : \quad V_h f(x) = f(x + h)$$

Then it follows $U_{A^n} V_h U^*_{A^n} = V_{\tau_{A^n}(h)}$

so that they induce a unitary representation

$$[A^n, h] \to V_n U_{A^n}$$

of $G_1 \overset{\tau}{\otimes} G_2$ on the space $L^2(T^n)$.

Clearly V is faithful, and hence any unitary representation of $G_1 \overset{\tau}{\otimes} G_2$ induces a unitary representation of G_1 and G_2 with the transversal commutation relation.

Since any continuous group automorphism on the torus can be associated with a unimodular integral matrix, we regard A as a unimodular integral matrix.

Let γ be an eigenvector of A with respect to a real eigenvalue λ (we assume their existence) and let $\{h_s\}$ be an 1-parameter subgroup of T^n defined by

Sem. on Probab.
Vol.36 1972年
P1-89

$$(x, h_s) = \exp\{i < r, x > s\}$$

Then the 1-parameter subgroup $\{h_s\}$ of T^n induces a 1-dimensional transversal group $\{V_s\}$ of U_A by

$$V_s f(x) = f(x + h_s)$$

with the commutation relation

$$U_A V_s = V_{\lambda s} U_A .$$

The constructions of transversal flows of A and its applications to isomorphy problem of group automorphisms on the torus will be given in § III.

Example 2. We put

G_1 : the multiplicative group of all positive numbers.

G_2 : the additive group of all real numbers.

τ : $\tau_g(h) = gh \qquad g \in G, \quad h \in G_2$.

Then $G = G_1 \overset{\tau}{\otimes} G_2$ is a 2-dimensional affine group. To find all the irreducible unitary representations of the group G is equivalent to find all the positive definite functions on G and to decompose them into elementary positive definite functions. The general form of the irreducible unitary representations of the group G obtained by I. Gelfand and M. Neumark will be given in § 2.5.

Let $(\Omega, \mathcal{B}, \mu, T_t)$ be a measurable flow on a probability space $(\Omega, \mathcal{B}, \mu)$ and its transversal flow $\{Z_s\}$ with expanding coefficient λ. They induce unitary operators $\{U_t\}$, $\{V_s\}$ on a Hilbert space $\mathcal{H}$.

$$U_\alpha f(x) = f(T_{-\frac{1}{\lambda}\log\alpha}\, x)$$

Sem. on Probab.
Vol.36 1972年
P1-89

$$V_s f(x) = f(Z_s x)$$

such that $U_\alpha V_s U_\alpha^* = V_{\alpha s}$

Thus $[\alpha, s] \to U_\alpha V_s$

is a unitary representation of G .

The spectral type of $\{U_\alpha\}$ is uniform Lebesgue, and $\{V_s\}$ is weakly mixing; these facts will be proved in § 4.2. The geodesic flow $\{T_t\}$ on a compact manifold with negative constant curvature and the horocycle flow $\{Z_s\}$ on it yield such an example.

Refer also to the flow of white noise in § 4.3.

Example 3. Let $G_1 = GL(R, n)$, $G_2 = R^n$ and $\tau_A(h) = Ah$, $A \in G_1$, $h \in G_2$

We call the subgroup $\{\Pi[e_1, h]\}$ the n-dimensional transversal group of the group $\{\Pi[g, e_2]\}$ where Π is a unitary representation of the group $G = G_1 \tilde{\otimes} G_2$.

The following two examples come from the relativistic quantum theory.

Example 4. Let G_1 be the proper Lorentz group, G_2 the space of space-time and

$$\tau_A(x) = Ax , \quad A \in G_1 , \quad x \in G_2 .$$

The group $G = G_1 \tilde{\otimes} G_2$ is the Poincaré group.

Example 5. Let

$G_1 = G$, the Poincaré group

$G_2 = \mathcal{S}(R^4)$, the Schwartz's $\mathcal{S}$-space

$(\tilde{\tau}_{[A,x]}(h))(y) = h(A^{-1}(y - x))$ (the inner product)

$x, y \in R^4$, $[A, x] \in G_1$, $h \in G_2$

Sem. on Probab.
Vol.36 1972年
P1-89

and put

$$\tau_{[A,x]} = (\tau_{[A,x]}^{-1})^{*}, \quad \text{the dual of } \tau_{[A,x]}^{-1} .$$

Put $G_1 \ni [A, x] \to U_{[A,x]}$

, where $U_{[A,x]}f(y) = f(A^{-1}(y - x))$

and then it is a unitary representation of G_1 on the Hilbert space $L^2(R^4, d\ )$.

Let $G_2 \ni f \to \varphi(f)$ be the representation by the field operator. Then $G = G_1 \overset{\tau}{\otimes} G_2 \ni ([A, x], f) \to \varphi(f)U_{[A,x]}$

is a unitary representation and

$$U_{[A,x]}\varphi(f) = \varphi(\tau_{[A,x]}f)U_{[A,x]} .$$

§ 2.4. The uniquencess of U with respect to V.

Let U and V be unitary representations of group G_1, and G_2 on the Hilbert space $\mathcal{H}$ respectively such that $U_g V_h = V_{t_g(h)} U_g$. Denote by $\mathcal{V}$ a von Neumann algebra generated by $\{ V_h ; h \in G_2 \}$ and denote W^g an automorphism of $\mathcal{V}$ obtained by extending (to $\mathcal{V}$) continuously and linearly a mapping

$$V_h \rightarrow U_g V_h U_g^* .$$

Then $\{ W^g ; g \in G_1 \}$ is a group of automorphisms of $\mathcal{V}$. Our problem, now, is to classify the set of all automorphisms of $\mathcal{V}$ into the equivalent classes.

In the followings, we assume G_1 and G_2 are locally compact abelian groups.

<u>Proposition 2.4.1.</u> Suppose $\mathcal{V}$ is maximal abelian and let U and U be unitary representations of G_1 such that $U_g V_h = V_{\tau_g(h)} U_g$ and $U'_g V_h = V_{\tau_g(h)} U'_g$. Then

$$U'_g = \alpha(g) U_g , \quad g \in G_1 ,$$

where $\alpha = \alpha(g)$ is a character of G_1.

<u>Proof.</u> $(U'_g U_g^{-1}) V_h = U'_g V_{\tau_g^{-1}(h)} U_g^{-1} = V_{\tau_{gg^{-1}}(h)} U'_g U_g^{-1} = V_h U'_g U_g^{-1}$ for any $h \in G_2$, and since $\mathcal{V}$ is maximal abelian,

$$U'_g U_g^{-1} = \alpha(g) I , \quad I \text{ is the identity.}$$

Sem. on Probab.
Vol.36 1972年
P1-89

Since U_g is unitary, $|\alpha(g)| = 1$. Moreover

$$U'_{g_1 g_2} = \alpha(g_1 g_2) U'_{g_1 g_2}$$
$$= U'_{g_1} U'_{g_2} = \alpha(g_1)\alpha(g_2) U'_{g_1} U_{g_2}$$

and hence $\alpha \in G_1^{\wedge}$, the character group of G_1. More generally, we have

Proposition 2.4.1. Let V and V' be unitary representation of G_2 on a Hilbert space $\mathcal{H}$, and we assume $\mathcal{V}$ is maximal abelian. Let U and U' be unitary representations of G_1 on $\mathcal{H}$ such that

$$U_g V_h U_g^* = V_{\tau_g(h)}$$
$$U'_g V'_h U'^*_g = V'_{\tau_g(h)}$$

Suppose V and V' are equivalent, say

$$WVW^* = V'$$

by a unitary operator W. Then

$$WU'_g W^* = \alpha(g) U_g, \qquad \alpha \in G_1^{\wedge}$$

Lemma 2.4.1. Suppose $\mathcal{V}$ is maximal abelian and let f_0 be the cyclic vector of V. Then there exists uniquely a unitary representation W such that

1) $W_g V_h W_g^{-1} = V_{\tau_g(h)}$

2) $W_g f_0 = f_0$

Sem. on Probab.
Vol.36 1972年
P1-89

Proof. Define a transformation W_g by

$$W_g V_h f_0 = V_{\tau_g(h)} f_0 , \quad h \in G_2$$

and we denote by the same letter W_g the continuous linear extension of W_g on the space $\mathcal{H}$. Plainly W_g is a unitary operator for any $g \in G_1$. The condition 2) is clear to see and 1) is deduced from the followings,

$$W_g(\sum \lambda_k V_{h_k} f_0) = \sum \lambda_k V_{\tau_g(h_k)} W_g f_0$$

and then

$$W_g V_h W_{g^{-1}}(\sum \lambda_k V_{h_k} f_0) = \sum \lambda_k V_{\tau_g(h\, \tau_{g^{-1}}(h_k))} W_g f_0$$

$$= V_{\tau_g(h)}(\sum \lambda_k V_{h_k} f_0)$$

Theorem 2.4.1. Let U and V are unitary representations of G_1 and G_2 on $\mathcal{H}$ respectively such that

$$U_g V_h U_g^* = V_{\tau_g(h)}$$

for some τ , and suppose $\mathcal{V}$ is maximal abelian. Then there exists 1-1 correspondence between the unitary representations U's of G_1 on $\mathcal{H}$ and the characters α_U G via the equation

$$U_g = \alpha_U(g) W_g .$$

Proof. It follows from the proposition 2.4.1 and the lemma 2.4.1

§2.5. Unitary representations of the transversal commutation relation.

Sem. on Probab.
Vol.36 1972年
P1-89

In this §, we shall give determine all the irreducible unitary representations of the transversal commutation relation $R^{+} \overset{\tau}{\otimes} R$ in a separable Hilbert space. The results and proofs in this § were obtained by I. Gelfand and M. Naimark [24]. They considered a group G of linear transformations

$$y = \alpha x + \beta$$

of the real axis, where α runs over all positive numbers and β runs over all real numbers. The group G contains two subgroups G_1 and G_2 generated by the transformations. $T_\alpha : x \to T_\alpha x = \alpha x$ and $Z_\beta : x \to Z_\beta x = x + \beta$, respectively. They clearly satisfy the commutation relations.

$$T_{\alpha_1} T_{\alpha_2} = T_{\alpha_1 \alpha_2} \tag{2.5.1}$$

$$Z_{\beta_1} Z_{\beta_2} = Z_{\beta_1 + \beta_2} \tag{2.5.2}$$

$$Z_\beta T_\alpha = T_\alpha Z_{\alpha\beta} \tag{2.5.3}$$

When we put $T_{e^{-\lambda t}} = S_t$ for some non zero λ, then the two relations (2.5.1) and (2.5.3) become

$$S_{t_1} S_{t_2} = S_{t_1 + t_2} \tag{2.5.4}$$

$$Z_\beta S_t = S_t Z_{\beta e^{-\lambda t}} \tag{2.5.5}$$

respectively.

Sem. on Probab.
Vol.36 1972年
P1-89

In the followings we prove the toeorem .

<u>Theorem 2.5.1.</u> Every irreducible unitary representation of G is eqivalent to one of the following types :

I. $\mathfrak{H}$ is 1-dimensional; $V_0 \equiv I$, U_α is the character of the multiplicative group of positive numbers.

II. $\mathfrak{H}$ consists of all square summable functions $f(x)$, $-\infty < x < \infty$ which are boundary values on the real axis of functions analytic in the upper half-plane:

$$V_\beta f(x)=f(x+\beta) \; ; \quad U_\alpha f(x)= \sqrt{\alpha}\, f(\alpha x)(W_t f(x)=e^{\frac{-\lambda}{2}t} f(e^{-\lambda t}x))$$

III. $\mathfrak{H}$ consists of all square summable functions $f(x)$, $-\infty < x < \infty$, which are the boundary values on the real axis of functions analytic in the lower half plane;

$$V_\beta f(x)=f(x+\beta) \; ; \quad U_\alpha f(x)= \sqrt{\alpha}\, f(\alpha x)(W_t f(x)=e^{-\frac{\lambda}{2}t} f(e^{-\lambda t}x))$$

Now suppose we are given a unitary representation π of G in a separable Hilbert space $\mathfrak{H}$ and denote U_α and V_β the unitary operators $\pi(T_\alpha)$ and $\pi(Z_\beta)$ respectively.

Denote by N the set of all invariant elements of $\mathfrak{H}$ under V_β , i.e.

$$N = \{ h \in \mathfrak{H} \mid V_\beta h = h \}$$

Since $V_\beta U_\alpha N = U_\alpha V_{\alpha\beta} N = U_\alpha N$, it follows $U_\alpha N = N$ so that we may reduce the representation π onto the subspace $M = N^\perp$, the orthogonal complement of N . Hereafter we confine ourselves to the space M

Let

$$V_\beta = \int_{-\infty}^{+\infty} e^{i\lambda\beta}\, dE_\lambda \qquad (2.5.6)$$

be the Stone decomposition of the 1-parameter group of unitary operators

Sem. on Probab.
Vol.36 1972年
P1-89

$\{V_\beta\}$, and by Hellinger-Hahn decomposition, the space M can be represented as the direct sum of the subspaces M_k which are spaned by the sets $\{E_\lambda h_k;\ -\infty < \lambda < +\}$ where $\{h_k\}$ is an orthonormal system of M. The probability measure $\sigma_k(\cdot)$ defined by

$$d\sigma_k(\lambda) = d(E_\lambda h_k, h_k) \qquad (2.5.7)$$

is absolutely continuous with respect to $\sigma_{k-1}(\cdot)$, ..., $\sigma_1(\cdot)$. The subspace M_k is equivalent to the space $L^2(\sigma_k)$ via the mapping $L^2(\sigma_k) \ni f \longrightarrow \int_{-\infty}^{\infty} f(\lambda)dE_\lambda h_k \in M_k$. Let P_k be the projection operator on the subspace M_k. Put $P_k f = f_k$ and we regard f_k as the function $f_k(\lambda) \in L^2(\sigma_k)$ by the above mapping. We shall write $f \sim \{f_k(\lambda)\}$. Then we can see easily $V_\beta f \sim \{e^{i\lambda\beta} f_k(\lambda)\}$

Put $P_k U_\alpha P_j = U_{kj}(\alpha)$. Then $U_{kj}(\alpha)$ is a bounded operator from M_j onto M_k, such that

$$U_\alpha f \sim \{\sum_j U_{kj}(\alpha) f_j(\lambda)\}$$

Denote by $\pi_{kj}(\alpha;\lambda)$ the funciton in $L^2(\sigma_k)$ corresponding to the element $P_k U_\alpha P_j h_j \in M_k$ i.e.

$$P_k U_\alpha P_j h_j = \int_{-\infty}^{+\infty} \pi_{kj}(\alpha;\lambda) dE_\lambda h_k .$$

Then, since $U_\alpha^* E_\lambda U_\alpha = E_{\lambda/\alpha}$, we get

$$P_k U_\alpha P_j f = P_k U_\alpha \int_{-\infty}^{+\infty} f_j(\lambda) dE_\lambda h_j$$

$$= \int_{-\infty}^{+\infty} f_j(\lambda) dE_{\lambda/\alpha} P_k U_\alpha h_j = \int_{-\infty}^{+\infty} f_j(\lambda/\alpha)\pi_{k_j}(\alpha : \lambda) dE_\lambda h_k$$

$$\sim \pi_{kj}(\alpha ; \lambda) f_j(\lambda/\alpha) . \qquad (2.5.8)$$

From the unitary condition of U_α , it follows

$$\sum_k \int_{-\infty}^{+\infty} |f_k(\lambda)|^2 \, d\sigma_k(\lambda) = \sum_k \int_{-\infty}^{+\infty} \sum_j \pi_{kj}(\alpha ; \lambda) f_j(\lambda/\alpha)|^2 \, d\sigma_k(\lambda) \qquad (2.5.9)$$

Putting in (2.5.9)

$$f_k(\lambda) = \begin{cases} 0 & (k \neq 1) \\ \chi_\Delta(\lambda) (k=1), & \text{the characteristic function of a set } \Delta \end{cases}$$

we get

$$\sigma_1(\Delta) = \sum_k \int_{\alpha\Delta} |\pi_{k1}(\alpha ; \lambda)|^2 \, d\sigma_k(\lambda)$$

and hence

$$\sigma_1(\alpha\Delta) = \sum_k \int_\Delta |\pi_{k1}(\tfrac{1}{\alpha} ; \lambda)|^2 \, d\sigma_k(\lambda) .$$

Thus $\sigma_1(\alpha\Delta)$ is absolutely continuous with respect to $\sigma_1(\Delta)$ because every σ_k is absolutely continuous with respect to σ_1 . It follows that there exist the density functions $\omega_k(\lambda)$ of σ_k with respect to the Lebesgue measure (k=1, 2, ...) ;

$$\sigma_k(\Delta) = \int_\Delta \omega_k^2(\lambda) |\lambda|^{-1} d\lambda \quad (k=1, 2, \ldots) \quad (2.5.10)$$

Put $E_k = \{\lambda \mid \omega_k(\lambda) = 0\}$, then $E_k \subset E_{k+1}$ and, since α runs over all positive numbers, E_1 is one of the sets (0) , $(0, +\infty)$, $(-\infty, 0)$. Now we define the mapping $L^2(\sigma_k) \ni f_k(\lambda) \longrightarrow \varphi_k(\lambda) = \omega_k(\lambda) f_k(\lambda) \in \hat{M}_k$, where $\hat{M}_k$ is the space of functions

Sem. on Probab.
Vol.36 1972年
P1-89

$\varphi(\lambda)$ with finite norm $\|\varphi\|^2 = \int |\varphi(\lambda)|^2 |\lambda|^{-1} d\lambda$ and with the property $\varphi(\lambda) = 0$ on E_k. Then it is an isometric onto mapping of $L^2(\sigma_k)$ onto M_k. We put $\hat{M} = \sum_{k=1}^{n_0} \oplus \hat{M}_k$, where n_0 is the number of the M_k and let $S_{k_j}(\alpha)$ be the operator on $\hat{M}$ corresponding to $U_{k_j}(\alpha)$. Then we get

$$U_{k_j}(\alpha) f_j \sim \pi_{k_j}(\alpha ; \lambda) f_j(\lambda/\alpha)$$

$$= \pi_{k_j}(\alpha ; \lambda) \omega_k(\lambda/\alpha)^{-1} \omega_k(\lambda) \omega_k(\lambda/\alpha) f_j(\lambda/\alpha)$$

$$= \pi'_{k_j}(\alpha ; \lambda) \varphi_j(\lambda/\alpha) \quad \text{for} \quad \lambda/\alpha \bar{\in} E_j$$

$$S_{k_j}(\alpha)\varphi_j(\lambda) = \pi'_{k_j}(\alpha ; \lambda) \varphi_j(\lambda/\alpha) \quad \text{for} \quad \lambda/\alpha \bar{\in} E_j$$

We don't define π'_{k_j} for $\lambda/\alpha \in E_j$

If we write $f \sim \varphi(\lambda) = \{\varphi_k(\lambda)\}$, then

$$\|f\|^2 = \sum_k \int_{-\infty}^{+\infty} |\varphi_k(\lambda)|^2 |\lambda|^{-1} d\lambda$$

$$= \sum \int_{-\infty}^{\infty} \Big| \sum_j \pi'_{k_j}(\alpha, \alpha\lambda) \varphi_j(\lambda)\Big|^2 |\lambda|^{-1} d\lambda$$

$$V_\beta f \sim e^{i\lambda\beta} \varphi(\lambda) ; \quad U_\alpha f \sim (\alpha ; \lambda) \varphi(\lambda/\alpha)$$

where $A(\alpha ; \lambda)\varphi(x) = \{ \sum_j \pi'_{k_j} \varphi_j(\lambda) \}$ (2.5.11)

For a.e. $\lambda/\alpha \bar{\in} \cup E_k$, $A(\alpha ; \lambda)$ is an isometric operator with domain $\hat{M}$, and for a.e. $\lambda/\alpha \in E_k$ with the domain $\sum_{i=1}^{k} \oplus \hat{M}_j$. The range of $A(\alpha ; \lambda)$ is $\hat{M}$ for a.e $\lambda/\alpha \bar{\in} E_k$

and $\sum_{j=1}^{k} \oplus \hat{M}_j$ for a.e, $\lambda \in E_k$.

Let $\lambda \in E_k \cap (\bigcup \alpha E_j)^c$, then $\hat{M}$ is transformd onto $\sum_{j=1}^{k} \oplus \hat{M}_j$ isometrically by $A(\alpha; \lambda)$. This is impossible, since $\sum_{j=1}^{k} \oplus \hat{M}_j$ is the finite dimensional subspace of $\hat{M}$. Therefore, for $\lambda \in \bigcup \alpha E_k$, λ must not belong to $\bigcup E_k$ and hence $\bigcup E_k$ is one of the sets (0), $(-\infty, 0)$, $(0, +\infty)$ .. Similarly we see the same is also true for each E_k .

Since $S_{\alpha_1} S_{\alpha_2} f = S_{\alpha_1} S_{\alpha_2} f \sim A(\alpha_1; \lambda) A(\alpha_2; \lambda/\alpha_1) \varphi(\lambda/\alpha_2\alpha_1)$

$$= A(\alpha_1 \alpha_2; \lambda) \varphi(\lambda/\alpha_1 \alpha_2) ,$$

we get

$$A(\alpha_1; \lambda) A(\alpha_2; \lambda/\alpha_1) = A(\alpha_1 \alpha_2; \lambda) \quad \text{a.e} \quad (\alpha_1, \alpha_2, \lambda) \tag{2.5.12}$$

Put $B(\alpha, \beta) = A(\alpha/\beta ; \alpha)$. Then the operator $\mathcal{B}(\alpha, \beta)$ is isometric and defined for a.e $(\alpha/\beta, \alpha)$ and hence, by Fubini's theorem, also for a.e (α, β) . The equation (2.5.12) is turned into the form

$$\mathcal{B}(\alpha, \beta) \mathcal{B}(\beta, \gamma) = \mathcal{B}(\alpha, \gamma) \quad \text{for a.e} \quad (\alpha, \beta, \gamma) \tag{2.5.13}$$

Then there exists a $\beta = \beta_0$ such that

$$\mathcal{B}(\alpha, \beta_0) \mathcal{B}(\beta_0, \gamma) = \mathcal{B}(\alpha, \gamma) \quad \text{for a.e} \quad (\alpha, \gamma) .$$

Put $\mathcal{B}(\alpha, \beta_0) = \mathcal{B}(\alpha)$, $\mathcal{B}(\beta_0, \gamma) = \mathcal{C}(\gamma)$; then by (2.5.13)

$$\mathcal{B}(\lambda) \mathcal{C}(\lambda/\alpha_1) \mathcal{B}(\lambda/\alpha_1) \mathcal{C}(\lambda/\alpha_1 \alpha_2)$$

$$= \mathcal{B}(\lambda, \lambda/\alpha_1) \mathcal{B}(\lambda/\alpha_1, \lambda/\alpha_1 \alpha_2)$$

Sem. on Probab.
Vol.36 1972年
P1-89

$$= \beta(\lambda, \lambda/\alpha_1\alpha_2) = \beta(\lambda)C(\lambda/\alpha_1\alpha_2)$$

Hence $C(\lambda)\beta(\lambda) = 1$ for a.e. λ so that

$$A(\alpha, \lambda) = \beta(\lambda)C(\lambda/\alpha) = \beta(\lambda)\beta^{-1}(\lambda/\alpha) .$$

Now put $\psi(\lambda) = \beta^{-1}(\lambda)\varphi(\lambda)$ and write $f \sim \psi(\lambda)$. Then

$$\| f \|^2 = \int_{-\infty}^{+\infty} \| \psi(\lambda) \|^2 \frac{d\lambda}{|\lambda|} = \sum_{k=1}^{n_0} \int_{-\infty}^{+\infty} \frac{|\psi_k(\lambda)|^2}{|\lambda|} d\lambda,$$

$$V_\beta f \sim e^{i\lambda\beta}\psi(\lambda) = \{ e^{i\lambda\beta} \psi_k(\lambda) \}$$

$$U_\alpha f \sim \psi(\lambda/\alpha) = \{ \psi_k(\lambda/\alpha) \}$$

Now we suppose the representation is irreducible; then M or N is (0) . In the first case, U_α is the character of the multiplicative group of positive numbers. In the second case, $\hat{M}$ is one-dimensional so that $\psi(\lambda)$ must be a scalar valued function. In this case the representation is irreducible if and only if $\hat{M}$ consists of functions $\psi(\lambda)$ vanishe on positive or on negative part of the real axis. Passing to the Fourier-transform

$$f(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\psi(\lambda)}{\sqrt{|\lambda|}} e^{i\lambda x} d\lambda ,$$

we obtain the theorem by the Paley-Wiener theorem.

Sem. on Probab.
Vol.36 1972年
P1-89

III The isomorphy problem for the group automorphisms and the transversal flows

§ 3.1. Outline of this chapter

In ergodic theory, there is an isomorphy problem which asks when two metrical automorphisms on a Lebesgue space are metrically isomorphic. This problem for automorphisms has been discussed by introducing several kinds of metric invariants. Since A. Kolmogorov, by using an invariant, entropy, has shown that spectral type is not necessarily a complete metric invariant for general automorphisms, it was conjectured that entropy is a complete metric invariant for K-automorphisms. Recently D. Ornstein has obtained the results that entropy is generally not complete invariant for K-automorphisms but it is a complete invariant for very weak Bernoulli automorphisms.

In this section, we shall show that the multi-dimensional transversal flows for a group automorphism on the finite diminsional torus play also a role of a complete metric invariant for group automorphisms on the torus. Our method gives us some informations 'how two group automorphisms are isomorphic' as can be seen later.

Let M_n be the n-dimensional torus and let A be a contiouous group automorphism on M_n. This mapping A becomes a metrical automorphism on the measure space with which the Haar measure and topological Borel field are associated.

We are concerned with the two types of equivalences, metrical and algebraic ones ; we say two group automorphisms A_1 and A_2 are

Sem. on Probab.
Vol.36 1972年
P1-89

metrically equivalent if there exists a measure preserving, 1-1 mapping σ such that $\sigma^{-1}A_1\sigma\omega = A_2\omega$ for a.e. $\omega \in M_n$. Group automorphisms A_1 and A_2 are said to be algebraically equivalent if there exists a continuous group automorphism θ on M_n such that $\theta^{-1}A_1\theta = A_2$.

Algebraic equivalence implies metrical one, but the converse implication is false, and so entropy is not complete invariant among algebraic equivalent classes ; our invariant, ergodic transversal flow with discrete spectrum may serve to the above two types of equivalence problems.

The notion of 1-parameter transversal flow was already introduced by ya. G. Sinai [5] and he showed a sufficient condition for the existence of it. In general, there does not always exist ergodic 1-parameter transversal flow of translation type and we, therefore, proceed to extend the notion of 1-parameter transversal flow into multi-dimentional one to ensure the existence of ergodic one.

For given two group automorphisms A_1 and A_2 on M_n, we can easily construct ergodic transversal flows with discrete spectrum, $\{Z_s^{(1)}\}$ and $\{Z_s^{(2)}\}$, but it should be noted that it is difficult to construct them in such a way that

$$A_k Z_s^{(k)} = Z_{Ts}^{(k)} A_k , \quad k=1, 2 ,$$

that is, the matrix T is common for two flows. On the while, in the case of algebraic equivalence, it is sufficient to seek such pair of transversal flows among the class of transversal flows of translation type, and eigenvalues of A_1 and A_2 give us much informations to construct them.

Sem. on Probab.
Vol.36 1972年
P1-89

§ 3.2. Definitions and Notations

In this §, we shall give, mainly, the definition of multi-dimensional transversal flow for metric automorphism on a probability space.

Definition 3.2.1 Let $(\Omega, \mathcal{B}, \nu)$ be a probability space. Let $\{Z_s\}$ be a family of metric automorphisms on Ω with parameter s which runs over m-dimensional Euclidean space R^m. Then we call $\{Z_s\}$ is an m-parameter flow, if it satisfies the group property,

1) $Z_{s+t} = Z_s Z_t \pmod 0$, $s, t \in R^m$

and the measurability condition,

2) the mapping, $(s, \omega) \longrightarrow Z_s\omega$, is measurable.

Definition 3.2.2 Denote by V_s the unitary operator on $L^2(\Omega, \nu)$ induced by Z_s;

$$V_s f(\omega) = f(Z_s \omega), \qquad f \in L^2(\Omega, \nu).$$

The m-parameter flow $\{Z_s\}$ is said to be ergodic, if any invariant function of $\{V_s\}$ is a constant function only. If there exists a complete orthonormal system $\{f_n\}$ and a set $\{\xi_n\}$ in R^m such that

$$V_s f_n = e^{i \langle s, \xi_n \rangle} f_n \qquad s \in R^m,$$

then we say the m-parameter flow $\{Z_s\}$ has the discrete spectrum $\{\xi_n\}$, where $\langle \cdot , \cdot \rangle$ means the inner product in R^m.

Sem. on Probab.
Vol.36 1972年
P1-89

<u>Definition 3.2.3</u> Let A be a metrical automorphism on Ω. An m-parameter flow $\{Z_s\}$ is <u>the transversal flow of A</u> if there exists a regular $m \times m$ matrix T such that

$$AZ_s = Z_{Ts}A, \qquad s \in R^m .$$

Although we defined an multi-dimensional transversal flow on a general probability space Ω, in the following through, we consider only the case where Ω is the n-dimensional torus M_n. The general cases are commented shortly in § 3.6.

Let M_n be the n-dimensional torus and ν be the normalized Haar measure of M_n. Then a continuous group automorphism on M_n can be considered as a metrical automorphism on the probability space (M_n, ν). It is well known that a group automorphism A on the torus $M_n = R^n/N^n$ is associated with a unimodular matrix which we also denote by the same letter A, if no confusion occurs. Let $\hat{M_n}$ be the characeter group of M_n. The elementasof M_n and that of $\hat{M_n}$ are denoted by $g, h, \ldots$ and $\hat{g}, \hat{h}, \ldots$, respectively.

We cite the well known theorems of P. Halmos and von Neumann [5] which are concerned with the ordinary 1-parameter ergodic flow with discrete spectrum. These theorems still hold in case of ergodic m-parameter flow. We proceed, without the proof, to the followings.

<u>Theorem A.</u> Every proper value of an ergodic m-parameter flow is simple, and the set of all proper values forms an additive group. Moreover the family of all eigenfunctions multiplied by suitable constants forms a group under the multiplication as function.

Sem. on Probab.
Vol.36 1972年
P1-89

Theorem B. Two ergodic m-parameter flows with discrete spectrums are metrically equivalent if and only if they are spectrally equivalent.

We use the above two theorems in the following.

§ 3.3. The existence of a transversal flow.

The following is an extension of Ya. G. Sinai's result [5; § 7] .

Lemma 3.3.1 Let A be a group automorphism on M_n, and suppose that there exist a regular $m \times m$ matrix T and φ, a homomorphic imbedding of $\hat{M}_n$ into R^m such that

$$\varphi(A^* g^\wedge) = T^*[\varphi(g^\wedge)] , \quad g^\wedge \in \hat{M}_n , \qquad (3.3.1)$$

where A^* and T^* denote the transposes of matrices A and T, respectively. Then there exists an m-parameter transversal flow $\{Z_s\}$ of A with discrete spectrum. If, in particular, φ is an isomorphism, then $\{Z_s\}$ is ergodic.

Proof. We define $\{g_s\}$ by

$$(g_s, g^\wedge) = \exp\{ i < s, \varphi(g^\wedge) > \} , \qquad (3.3.2)$$

then $\{g_s\}$ forms an m-parameter subgroup of M_n. Define Z_s by

$$Z_s g = g + g_s, \quad g \in M_n , \qquad (3.3.3)$$

then $\{Z_s\}$ is an m-parameter flow. Moreover we have

$$(Ag_s, g^\wedge) = (g_s, A^* g^\wedge) = \exp\{ i < s, \varphi(A^* g^\wedge) > \}$$

$$= \exp\{ i < s, T^* \varphi(g) > \} = \exp\{ i < Ts, \varphi(g) > \}$$

$$= (g_{Ts}, g^\wedge)$$

Sem. on Probab.
Vol.36 1972年
P1-89

for any $g^\wedge \in M_n$, i.e., $Ag_s = g_{Ts}$, which implies that $AZ_s = Z_{Ts}A$.

The fact that $\{Z_s\}$ has discrete spectrum is deduced from the following ;

$$V_s g^\wedge(g) = g^\wedge(Z_s g) = g^\wedge(g+g_s) = \exp\{ i \langle s , \varphi(g^\wedge) \rangle \} \cdot \hat{g}(g) ;$$

this shows that the spectrum of $\{Z_s\}$ is just the set $\{ \varphi(g^\wedge) ; g^\wedge \in M_n \}$.

Suppose that φ be an isomorphism. Then the relation,

$$(g_s, g^\wedge) = \exp\{ i \langle s, \varphi(g^\wedge) \rangle \} = 1$$

for any g_s, implies $g^\wedge = 0$, therefore, the subgroup $\{g_s\}$ is dense in M_n. Let $f = \sum_{g^\wedge \in M_n^\wedge} C(g^\wedge) g^\wedge$ be the Fourier expansion of $f \in L^2(M_n)$. Suppose f be an invariant function of $\{V_s\}$, then

$$f = V_s f = \sum C(g^\wedge) e^{i \langle s, \varphi(g^\wedge) \rangle} \cdot \hat{g}$$

$$= \sum C(g^\wedge) g^\wedge .$$

This implies $C(g^\wedge) = 0$, unless $g^\wedge = 0$. Hence f is a constant function, i.e., $\{Z_s\}$ is ergodic.

Thus a triple $\{ m, \varphi, T \}$ yields a transversal flow of A. For convenience, we shall agree to say that a triple $\{ m, \varphi, T \}$ is <u>a solution of (3.3.1)</u>, if it satisfies (3.3.1). If, in particular, the flow $\{Z_s\}$ defined by (3.3.2) and (3.3.3) is ergodic, $\{ m, \varphi, T \}$ is said to be <u>an ergodic solution</u> of (3.3.1)

<u>Lemma 3.3.2</u> Any group automorphism A has an ergodic solution of (3.3.1).

Sem. on Probab.
Vol.36 1972年
P1-89

Proof. Let $e_k^\wedge = (\overbrace{0, 0, \ldots, 0}^{k}, 1, 0, \ldots, 0) \in M_n^\wedge$, $\varphi(e_k^\wedge) = e_k^\wedge \in R^n$, $(k=1, \ldots, n)$ and $A = T$. Then (3.1) has the ergodic solution $\{n, \varphi, A\}$.

Now we obtain the following existence theorem, which is an easy concequence from the previous two lemmas.

Theorem 3.3.1 For a group automorphism on M_n, there exist ergodic transversal flows with discrete spectrum.

We shall agree to say that the m-parameter flow $\{Z_s;\ Z_s g = g+g_s\}$ defined in (3.3) is of translation type.

Remark. There exist, in fact, group automorphisms which have no ergodic 1-parameter transversal flow of translation type, but have ergodic transversal flows with higher dimensional parameter. For example, the group automorphism associated with the unimodular matrix $\begin{pmatrix} 1 & 3 & 4 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{pmatrix}$ is such one.

Once eigenvalues of a matrix A are known, it is easy to form a tranaversal flow of A with discrete spectrum. To clearify this situation, we shall list several examples in the following.

Example 1. Suppose that the matrix A has a real eigenvalue λ and let $r = (r_1, r_2, \ldots, r_n)$ be an eigenvector corresponding to λ. Define a homomorphism φ from $M_n^\wedge$ into R^1 by

$$\varphi(e_k^\wedge) = r_k, \qquad k = 1, \ldots, n.$$

Then, by Lemma 3.1, the triple $\{1, \varphi, \lambda\}$ is a solution of (3.1)

Sem. on Probab.
Vol.36 1972年
P1-89

and it determines a transversal flow $\{ Z_s \}$ of A. Since φ is a homomorphism, $\{ Z_s \}$ is of the following form as was shown in the proof of Lemma 3.1,

$$Z_s g = g + g_s$$

(refer to Ya. G. Sinai [21]).

Example 2. Suppose that the matrix A has an eigenvalue $\lambda = \alpha + i\beta$ and let $r = (r_1, r_2, \ldots, r_n)$ be a corresponding eigenvector. We set $u = (\mathrm{Re}\, r_1, \ldots, \mathrm{Re}\, r_n)$ and $v = (\mathrm{Im}\, r_1, \ldots, \mathrm{Im}\, r_n)$. Then we have the relations, $Au = \alpha u - \beta v$, and $Av = \beta u + \alpha v$. Let $\varphi(g^\wedge) = \langle u, g^\wedge \rangle$ and $\psi(g^\wedge) = \langle v, g^\wedge \rangle$. Then $\tau(g^\wedge) = (\varphi(g^\wedge), \psi(g^\wedge))$ is a homomorphism from $M_n^\wedge$ into R^2. Denote by T the matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Then we obtain

$$\tau(A^* g^\wedge) = T^* \tau(g^\wedge), \qquad g^\wedge \in M_n^\wedge .$$

Therefore the triple $\{ 2, \tau, T \}$ is a solution of (3.1) so that it determines a 2-parameter transversal flow $\{ Z_{(s,t)} ; (s,t) \in R^2 \}$ of A.

Example 3. Consider an ergodic solution $\{ m, \varphi, T_1 \}$ of (3.1). Set $T_2 = ST_1S^{-1}$, where S is an $m \times m$ regular real matrix. Let $\psi(g^\wedge) = S\varphi(g^\wedge)$, $g^\wedge \in M_n^\wedge$. Then $\{ m, \psi, T_2 \}$ is also an ergodic solution of (3.1). Hence, if all eigenvalues of the matrix A is real and if we can find S such that $SAS^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, then we have an ergodic solution $\left\{ n, \psi, \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right\}$, where $\psi(e_k^\wedge) = Se_k^\wedge$,

Sem. on Probab.
Vol.36 1972年
P1-89

$k=1, .., n$. In this case, the ergodic transversal flow $\{ Z_{(s_1,..,s_n)} \}$ constructed by the solution has the following form ;

$$Z_{(s_1, .., s_n)} g = g + g_{(s_1, .., s_n)}$$

where $g_{(s_1, .., s_n)} = g_{s_1}^{(1)} + \cdots + g_{s_n}^{(n)}$ and $g_{s_k}^{(k)}$ is defined by the same way as in Example 1 with respect to λ_k , $k=1, .., n$.

In the proof of Lemma 3.2 , we have constructed a n-parameter ergodic transversal flow for group automorphism A on M_n . On the other hand, group automorphism may have another ergodic transversal flow with discrete spectrum of lower dimensional parameter than n . To get such transversal flow, we may well construct subgroups $\{ g_{s_1}^{(1)} \}, .., \{ g_{s_m}^{(n)} \}$ $(m \leqq n)$ corresponding to eigenvalues of A, $\lambda_1, .., \lambda_k, c_{k+1}, .., c_m$, respectively, where $\lambda_1, .., \lambda_k$ are reals, $c_{k+1}, .., c_m$ are complexes and $k+2(m-k) \leqq n$. Let s_j be 1 or 2 dimensional parameters according to $1 \leqq j \leqq k$ or $k+1 \leqq j \leqq m$. Then $k+2(m-k)$ dimensional flow

$$Z_{(s_1, .., s_m)} g = g + g_{s_1}^{(1)} + \cdots + g_{s_m}^{(m)}$$

satisfies the relation

$$AZ_{(s_1, .., s_m)} g = Z_{T(s_1, .., s_m)} Ag ,$$

where

$$T = \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & T_{k+1} & & \\ & & & & \ddots & \\ 0 & & & & & T_m \end{pmatrix} \quad \text{and} \quad T_j = \begin{pmatrix} \mathrm{Re}\, c_j & \mathrm{Im}\, c_j \\ -\mathrm{Im}\, c_j & \mathrm{Re}\, c_j \end{pmatrix}$$

Sem. on Probab.
Vol.36 1972年
P1-89

, $j = k+1, .., m$. It is easy to see that there exists some positive integer $k+2(m-k)$ ($\leq n$) such that the above transversal flow is ergodic.

§ 3.4. A metrical equivalence theorem for group automorphisms on M_n .

In this § , we give a theorem which asserts that a group automorphism on the torus is completely determined by its ergodic transversal flow with discrete spectrum.

Lemma 3.4.1. Let A be a group automorphism on M_n and $\{ Z_s \}$ be an ergodic m-parpmeter transversal flow of A with discrete spectrum $\Gamma = \{ \mu \}$ such that $AZ_s = Z_{Ts}A$. Then there exists a complete orthonormal system $\{ f_\mu \}$ in $L^2(M_n)$ with the following properties;

(1) f_μ is an eigenfunction of $\{ V_s \}$ corresponding to μ .

(2) $f_{\mu+\xi} = f_\mu f_\xi$, $\mu, \xi \in \Gamma$

(3) $U_A f_\mu = f_{T^*\mu}$,

where $V_s f(g) = f(Z_s g)$ and $U_A f(g) = f(Ag)$.

Proof. Let $\{ h_\mu ; \mu \in \Gamma \}$ be a basis of $L^2(M_n)$ each of which is an eigenfunction of $\{ V_s \}$. Since $\{ Z_s \}$ is ergodic, the absolute value of h_μ is a constant function, so we may assume $\{ h_\mu \} = 1$. By the relation, $V_s h_\mu h_\xi = e^{i\langle s, \mu+\xi \rangle} h_\mu h_\xi$ and the simplicity of spectrum of $\{ V_s \}$, we get $h_\mu h_\xi = C(\mu, \xi) h_{\mu+\xi}$, where $|C(\mu, \xi)| = 1$. Put $f_\mu = \overline{h_\mu(0)} h_\mu$, where $\overline{h_\mu(0)}$ is the complex conjugate of the value of h_μ at the eidentity 0 of M_n . It is trivial to see (1) . The relation (2) is deduced from the following;

Sem. on Probab.
Vol.36 1972年
P1-89

$$f_\mu f_\xi = \overline{h_\mu(0)h_\xi(0)h_\mu h_\xi} = \overline{C(\mu,\xi)h_{\mu+\xi}(0)C(\mu,\xi)h_{\mu+\xi}}$$
$$= f_{\mu+\xi} .$$

To obtain (3) , note that $U_A V_{Ts} = V_s U_A$ and

$$U_A V_s f_\mu = e^{i\langle s,\mu\rangle} U_A f_\mu = e^{i\langle T^{-1}s, T^*_\mu\rangle} U_A f_\mu = V_{T^{-1}s} U_A f_\mu .$$

Hence

$$U_A f_\mu(0) = cf_{T^*_\mu}(0) = f_\mu(A(0)) = f_\mu(0) = 1 ,$$

and we get, $c = 1$.

<u>Theorem 3.4.1.</u> Let A_1 and A_2 be group automorphisms on M_n .(*) Then they are metrically eqivalent if and only if there exist ergodic transversal flows $\{Z_s^{(1)}\}$ and $\{Z_s^{(2)}\}$ with discrete spectrum of A_1 and A_2 , respectively, which satisfy

(a) they are spectrally equivalent

(b) $A_k Z_s^{(k)} = Z_{Ts}^{(k)} A_k$, k=1, 2 .

<u>Proof.</u> Necesssity: Let σ be a metrical automorphism on M_n such that $\sigma^{-1}A_1\sigma = A_2$. By Theorem 3.1 , there exists an ergodic transversal flow $\{Z_s^{(1)}\}$ with discrete spectrum such that $A_1 Z_s^{(1)} = Z_{Ts}^{(1)} A_1$ Let $Z_s^{(2)} = \sigma^{-1} Z_s^{(1)} \sigma$, then we get

$$A_2 Z_s^{(2)} = (\sigma^{-1}A_1\sigma)(\sigma^{-1}Z_s\sigma) = \sigma^{-1}Z_{Ts}^{(1)}A_1\sigma = Z_{Ts}^{(2)}A_2 .$$

It is easy to see that $\{Z_s^{(2)}\}$ has the required properties.

Sufficiency: Denote the common discrete spectrum of $\{Z_s^{(1)}\}$ and $\{Z_s^{(2)}\}$ by $\Gamma = \{\mu_j, \ j=1, 2, \dots\}$. Then Γ forms a discrete

Sem. on Probab.
Vol.36 1972年
P1-89

additive group, and so the character group G of is a compact abelian group. Let $\{ f_\mu ; \mu \in \Gamma \}$ be a family of eigenfunctions of $\{ V_s^{(1)} \}$ which satisfies the conditions of (1)$\sim$(3) in Lemma 4.1 . For conve-ience, we rewrite μ by h_μ when μ is regarded as a function in $L^2(G)$; $h_\mu(\omega) = (\omega, \mu)$, $\omega \in G$. The mapping $f_\mu \longrightarrow h_\mu$ can be xtended to a unitary operator W_1 from $L^2(M_n)$ to $L^2(G)$. By Theorem , we see that W_1 is multiplicative, and therefore there exists a etrical isomorphism σ_1 from G into M_n such that

$$h_\mu(\omega) = W_1 f_\mu(\omega) = f_\mu(\sigma, \omega) , \quad \omega \in G , \quad j=1, 2, \ldots \quad .$$

et

$$\widetilde{A}_1 = \sigma_1^{-1} A_1 \sigma_1 \quad , \quad \widetilde{Z}_s^{(1)} = \sigma_1^{-1} Z_s^{(1)} \sigma_1 \quad , \quad \text{and} \quad (\omega_s, \mu) = \exp\{ i \langle s, \mu \rangle \}.$$

hen we get

$$\widetilde{A}_1 \widetilde{Z}_s^{(1)} = \widetilde{Z}_{Ts}^{(1)} \widetilde{A}_1$$

$$(\widetilde{Z}_s^{(1)} \omega , \mu) = h_\mu(\sigma_1^{-1} Z_s^{(1)} \quad _1 \quad) = f \quad (Z_s^{(1)} \quad _1 \quad)$$

$$= e^{i \langle s, \mu \rangle} f_\mu(\sigma_1 \omega) = (\omega_s, \mu)(\omega, \mu) = (\omega_s + \omega, \mu) ,$$

amely, $\widetilde{Z}_s^{(1)} = \omega + \omega_s$ -(4.1)

oreover we have

$$(\widetilde{A}_1 \omega_s, \mu) = h_\mu(\sigma_1^{-1} A_1 \sigma_1 \omega_s) = f_\mu(A_1 \sigma_1 \omega_s) = f_{T^* \mu}(\sigma_1 \omega_s)$$

$$= (\omega_s, T^* \mu) = e^{i \langle s, T^* \mu \rangle} = e^{i \langle Ts, \mu \rangle} = (\omega_{Ts}, \mu) .$$

ence we obtain $\widetilde{A}_1 \omega_s = \omega_{Ts}$ -(4.2)

e shall show that $\widetilde{A}_1$ is a group automorphism on G . We have

Sem. on Probab.
Vol.36 1972年
P1-89

$$\begin{aligned} h_\mu(\widetilde{A}_1(\omega+\omega')) &= f_\mu(\sigma_1\widetilde{A}_1(\omega+\omega')) = f_\mu(A_1\sigma_1(\omega+\omega')) \\ &= f_{T^*\mu}(\sigma_1(\omega+\omega')) = h_{T^*\mu}(\omega+\omega') \\ &= f_{T^*\mu}(\omega)h_{T^*\mu}(\omega') = f_{T^*\mu}(\sigma_1,\omega)f_{T^*\mu}(\sigma_1,\omega') \\ &= f_\mu(A_1\sigma_1\omega)f_\mu(A_1\sigma_1\omega') = h_\mu(\widetilde{A}_1\omega)h_\mu(\widetilde{A}_1\omega') \\ &= h_\mu(\widetilde{A}_1\omega+\widetilde{A}_1\omega'), \quad \text{for any} \quad \mu\in G^\wedge. \end{aligned}$$

By the same way as σ_1, we can definean isomorphism σ_2 from G into M_n, and using σ_2, we define a group automorphism $\widetilde{A}_2$ on G and its transversal flow $\{\widetilde{Z}_s^{(2)}\}$. Then we obtain the similar formulas to (4.1) and (4.2) ; $\widetilde{Z}_s^{(2)}\omega = \omega+\omega_s$ and $\widetilde{A}_2\omega_s = \omega_{Ts} = A_1\omega_s$

So we conclude, $\widetilde{A}_1 = \widetilde{A}_2$. Let $\sigma = \sigma_1\sigma_2^{-1}$, then we get $\sigma^{-1}A_1\sigma = A_2$

This completes the proof.

§ 3.5. An algebraic equivalence for group automorphisms on M_n.

As an application of the previous discussions, we shall give a necessary and sufficient condition of algebraic equivalence for group automorphisms on M_n.

Theorem 3.5.1. Let A_1 and A_2 be group automorphisms on M_n. Then they are algebraically equivalent if and only if there exist ergodic solutions $\{m, \varphi_k, T\}$ (k=1, 2) of

$$\varphi_k(A_k^* g^\wedge) = T\varphi_k(g^\wedge) \qquad -(3.5.1)$$

such that $\varphi_1(M_n^\wedge) = \varphi_2(M_n^\wedge)$.

Sem. on Probab.
Vol.36 1972年
P1-89

Proof. Let $\{ Z_s^{(k)} \}$ be ergodic transversal flow of A_k which is defined in the same way as in (3.3.3). Since $\{ Z_s^{(k)} \}$ is of translation type, all characters $g^\wedge \in M_n^\wedge$ are eigenvectors of $\{ Z_s^{(k)} \}$, and moreover the spectrum of it is just the set $\varphi_k(M_n)$; we denote the set $\varphi_1(M_n^\wedge) = \varphi_2(M_n^\wedge)$ by Γ. Let σ_1 and σ_2 be isomorphisms from $G = \Gamma^\wedge$ onto M_n which are defined in the proof of Theorem 4.1. Since σ_k is character preserving, namely, the function $\hat{g}(\sigma_k \omega)$ in $\omega \in G$ is a character of G, we sde that σ_k is a continuous group isomorphism from G onto M_n. Thus we get the continuous group automorphisms $\widetilde{A}_k = \sigma_k^{-1} A_k \sigma_k$ $(k=1, 2)$ such that $\widetilde{A}_1 \omega_s = \widetilde{A}_2 \omega_s$, $s \in R^m$. Since the subgroup $\{\omega_s\} \subset G$ is dense in G, we conclude $\widetilde{A}_1 = \widetilde{A}_2$, namely, $(\sigma_1 \sigma_2^{-1})^{-1} A_1 (\sigma_1 \sigma_2^{-1}) = A_2$. Clearly $\sigma_1 \sigma_2^{-1}$ is a continuous group automorphism on M_n.

As illustrated in Examples $1 \sim 3$, a solution $\{ m, \varphi, T \}$ can be constructed from the eigenvalues and eigenvectors of A. Note that it may happen that only one eigenvalue with eigenvector corresponding to it yields us an ergodic solution. We shall state about this case as a corollary.

Corollary. Group automorphisms A_1 and A_2 are algebraically equivalent if the unimodular integral matrices associated with them have eigenvectors $r_1 = (r_1^{(1)}, r_2^{(1)}, \ldots, r_n^{(1)})$ and $r_2 = (r_1^{(2)}, r_2^{(2)}, \ldots, r_n^{(2)})$ corresponding to an eigenvalue λ in common such that $\{ r_j^{(1)} \}$ are integrally independent, and $r_2 = Br_1$, where B is an unimodular integral matrix.

Sem. on Probab.
Vol.36 1972年
P1-89

§ 3.6. Some remarks

As was shown in § 3.2 , our invariant can be defined for not only group automorphisms on the torus but also a general class of metrical automorphisms on a probability space. It is still open when it exists for metrical automorphisms; a partial answer to this problem can be given as follows.

1. If a metrical automorphism A on a Lebesgue space has an ergodic m-parameter transversal flow with discrete spectrum, A is isomorphic to an automorphism (not necessarilly group automorphism) on a compact abelian group with which the Haar measure is associated.

2. Let G be a compact abelian group with the Haar measure and let G be its character group. Let A be a group automorphism on G and suppose that there exists φ , an isomorphic imbedding of G into R^m such that

$$\varphi(A^* g^\wedge) = T^* \varphi(g^\wedge) \quad , \quad g^\wedge \in G \, ,$$

where T is an $m \times m$ regular matrix. Then we can construct an ergodic m-parameter transversal flow of A in a similar way as in § 3.3.

Sem. on Probab.
Vol.36 1972年
P1-89

IV. Spectral type of 1-parameter group of unitary operators with transversal group.

It is an important problem to determine the spectral type of automorphisms or flows on a probability measure space. We shall deal with a unitary operator U and a 1-parameter group of unitary operators $\{U_t\}$ on a separable Hilbert space H, and discuss their spectral types, appealing to the transversal commitation relation.

For our purpose, a transversal group, if it exists, plays an important role. A 1-parameter group $\{V_s\}$ of unitary operators is said to be a transversal group for U, if it satisfies

$$UV_s = U_{\lambda s}U \quad \text{for some } \lambda \neq 0 .$$

Similarly, $\{V_s\}$ is a transversal group for $\{U_t\}$, if it satisfies

$$U_tV_s = V_{s\exp(\lambda t)}U_t \quad \text{for some } \lambda \neq 0 .$$

Ya. G. Sinai in [4] has already proposed the idea of a transversal field which is useful to see whether a given flow $\{T_t\}$ (an automorphism A) on a probability space Ω is Kolmogorov flow (Kolmogorov automorphism) or not.

For a flow $\{T_t\}$ (an automorphism A) on (Ω, P), if there exists an another flow $\{Z_s\}$ on (Ω, P) such that

$$T_tZ_s = Z_{u(s,t)}T_t \qquad (AZ_s = S_{u(s)}A) ,$$

then the transversal field can be constructed as the measurable partition of the measure space (Ω, P) along the orbits of $\{Z_s\}$ in finite time intervals. Such a flow $\{Z_s\}$ induces a 1-parameter

Sem. on Probab.
Vol.36 1972年
P1-89

group of unitary operatros $\{V_s\}$ on the space $L^2(\Omega)$ in a usual manner. The group $\{V_s\}$ satisfies the commutation relation mentioned above with $\{U_t\}$ induced by $\{T_t\}$ (with U induced by A) . In our approach, although the structure of measure space never contributes, the transversal commutation relation gives us many informations about the spectral type of U or $\{U_t\}$.

There are many important examples of unitary operators or 1-parameter groups of unitary operators with transversal groups such as flow of Brownian motion, geodesic flow on manifold of constant negative curvature and group automorphisms on n-dimensional torus. These examples can be investigated by our method as will be seen later.

§ 4.1. Definitions.

We are informed rather poor knowledge about the property of 1-parameter group of unitary operators $\{U_t\}$ (a unitary operator U) if we regard $\{U_t\}$ (U) only to be a strongly continuous unitary representation of the additive group of real numbers (integers) . We shall therefore introduce another 1-parameter group of unitary operators $\{V_s\}$ which transforms every orbit of $\{U_t\}$ (U) onto another.

Let H be a separable Hilbert space.

DEFINITION 4.1.1. A 1-parameter group of unitary operators $\{V_s\}$ on H is a transversal group with λ for a unitary operator U on H if they satisfy the commutation relation

(4.1.1) $$UV_s = V_{\lambda s}U , \quad -\infty < s < +\infty$$

for a nonzero real λ .

Sem. on Probab.
Vol.36 1972年
P1-89

DEFINITION 4.1.2. A 1-parameter group of unitary operators $\{V_s\}$ on H is a transversal group with λ for a 1-parameter group $\{U_t\}$, if they satisfy the relation

$$U_tV_s = V_{s\exp(\lambda t)}U_t, \quad -\infty < s,t < +\infty \tag{4.1.2}$$

for a nonzero real λ.

The relations (4.1.1) and (4.1.2) are the special cases of the followings ;

$$UV_s = V_{u(s)}U \tag{4.1.1'}$$

and

$$U_tV_s = V_{u(s,t)}U_t , \tag{4.1.2'}$$

respectively.

Observing the relation that V_s transforms U_t-orbits into themselves, (4.1.1)' and (4.1.2)' seem to be more reasonable. However the following propositions 4.1.1 and 4.1.2 enable us to make use of (4.1.1) and (4.1.2) to define a transversal group.

PROPOSITION 4.1.1. Let $u = u(s)$ be a nonzero aperiodic measurable function. Then the relation

$$UV_s = V_{u(s)}U$$

implies that

$$u(s) = \lambda s ,$$

for some $\lambda \neq 0$.

The proof follows from the assumptions and the following equalities

$$UV_{(s+t)} = UV_sV_t = V_{u(s)}UV_t = V_{u(s)}V_{u(t)}U = V_{(u(s)+u(t))}U = V_{u(s+t)}U .$$

Sem. on Probab.
Vol.36 1972年
P1-89

PROPOSITION 4.1.2. Suppose that V_s is aperiodic and $u = u(t,s)$ is a nonzero measurable function with respect to both t and s. Then the relation

$$U_t V_s = V_{u(t,s)} U_t$$

implies that

$$u(t,s) = \mu s \exp(\lambda t)$$

for some nonzero μ and λ.

Proof. From the similar calculation to the previous proposition, we see that $u(t,s) = sv(t)$, for some function $v(t)$. We have

$$U_{t_1+t_2} V_s = U_{t_1} V_{u(t_2,s)} U_{t_2} = V_{u(t_1,u(t_2,s))} U_{t_1+t_2}$$

$$= V_{u(t_1+t_2,s)} U_{t_1+t_2},$$

and concequently

$$u(t_1 + t_2,s) = sv(t_1 + t_2)$$

$$= u(t_1,u(t_2,s)) = u(t_2,s)v(t_1) = sv(t_1)v(t_2).$$

It follows that $v(t) = \mu \exp(\lambda t)$ for some $\mu \neq 0$ and $\lambda \neq 0$. This completes the proof.

Throughout this section, we assume that

(A) H is a separable Hilbert space,

and

(B) $N \equiv \{\phi \in H : V_s\phi = \phi \text{ for all } s\} = \{0\}$.

In the case where (B) is not fulfilled, the simultaneous reduction of $\{U_t\}$ (U) and $\{V_s\}$ onto $H \ominus N$ leads to our situation.

Sem. on Probab.
Vol.36 1972年
P1-89

§ 4.2. Spectral type of a unitary operator with transversal group.

In this section, we assume that U has a transversal group $\{V_s\}$ with $|\lambda| \neq 1$. The proof will be given only in the case where $|\lambda| > 1$. In case U has a transversal group $\{V_s\}$ with $|\lambda| > 1$, similar discussions lead us to the same results.

Let Γ be the set of all proper values of $\{V_s\}$, and for some positive μ_0, put $I_m^+ = [\mu_0/|\lambda|^m, \mu_0/|\lambda|^{m+1}]$ and $I_m^- = -I_m^+$ $(m = 0, \pm 1, \pm 2, \dots)$. Then we get a partition of Γ

$$\Gamma = \bigcup_{m=-\infty}^{+\infty} \{\Gamma_m^+ \cup \Gamma_m^-\}$$

where

$$\Gamma_m^+ = \Gamma \cap I_m^+$$

$$\Gamma_m^- = \Gamma \cap I_m^-$$

LEMMA 4.2.1. We have

(4.2.1) $\Gamma_m^{\pm} = \lambda^{-m}\Gamma_0^{\pm}$, if $\lambda > 0$, or if $\lambda < 0$ and m is even,

$\Gamma_m^{\pm} = \lambda^{-m}\Gamma_0^{\pm}$, if $\lambda < 0$ and m is odd.

Proof. Suppose $V_s\varphi = \exp(is\mu)\varphi$ for any s. Then, from (4.1.1) we get

$$V_s U^m \varphi = U^m V_{s\lambda^{-m}}\varphi = \exp(is\mu\lambda^{-m})U^m\varphi,$$

that is, $\mu\lambda^{-m} \in \Gamma$ $(m = 0, \pm 1, \pm 2, \dots)$, and it is easy to get the conclusion.

Denote the proper vectors by φ_n^k corresponding to $\mu_k \in \Gamma_0^+ \cup \Gamma_0^-$ ($k = 1,2, \ldots, n = 1,2, \ldots, n_k$, where n_k is the multiplicity of μ_k). Then, for all s,

(4.2.2) $$(U^m\varphi_n^k, U^{m'}\varphi_{n'}^{k'}) = (V_s U^m\varphi_n^k, V_s U^{m'}\varphi_{n'}^{k'})$$
$$= (U^m V_{s\lambda^{-m}}\varphi_n^k, U^{m'} V_{s\lambda^{-m'}}\varphi_{n'}^{k'})$$
$$= \exp[is(\mu_k \lambda^{-m} - \mu_{k'}\lambda^{-m'})](U^m\varphi_n^k, U^{m'}\varphi_{n'}^{k'})$$

and

$$(U^m\varphi_n^k, U^m\varphi_{n'}^k) = (\varphi_n^k, \varphi_{n'}^k) .$$

These equalities imply

$$U^m\varphi_n^k \perp U^{m'}\varphi_{n'}^{k'} ,$$

unless $\mu_k = \mu_{k'}$, $m = m'$ and $n = n'$ hold simultaneously.

Consequently, denoting $H_n^k = \mathfrak{S}\{U^m\varphi_n^k : m = 0, \pm 1, \ldots\}$, we have

(4.2.3) $$H_n^k \perp H_{n'}^{k'} \qquad (k \neq k' \text{ or } n \neq n')$$

<u>LEMMA 4.2.2</u> The operator U has simple Lebesgue spectrum on each cyclic subspace H_n^k.

<u>Proof.</u> Denoting

(4.2.4) $$U^m = \int e^{im\xi} dE^U(\xi) ,$$

and noticing (4.2.2), we have

$$\int e^{im\xi} d\,\| E^U(\xi)\varphi_n^k \|^2 = (U^m\varphi_n^k, \varphi_n^k) = 0$$

for any n,k and $m \neq 0$. Paley-Wiener's tehorem, therefore,

Sem. on Probab.
Vol.36 1972年
P1-89

implies the conclusion.

THEOREM 4.2.1. Suppose that for a unitary operator U on a separable Hilbert space H, there exists a transversal group $\{V_s\}$ with $|\lambda| \neq 1$ which has purely point spectrum. Then U has $(\sum n_k)$-multiple Lebesgue spectrum.

Proof. Denote by $M_m^+ (M_m^-)$ the subspace of H spanned by all proper vectors corresponding to $\Gamma_m^+ (\Gamma_m^-)$. Then, from (3.1), we get

$$H = \sum \oplus (M_m^+ \oplus M_m^-) .$$

While the equality $U^m M_n^+ = M_{n+m}^{\pm} (U^m M_n^{\pm} = M_{n+m}^{\mp})$, and the construction of H_n^k shows that

$$U^m M_0^+ (U^m M_0^-) \subset \sum \oplus H_n^k \subset H \tag{4.2.5}$$

for $m = 0, \pm 1, \ldots.$ This leads us to the conclusion

$$\sum_{n,k} \oplus H_n^k = H . \tag{4.2.6}$$

Hence, combining (3.6) with the lemma 3.2, we see that the spectral measure of U does not have singular component with respect to Lebesgue measure, and that H_n^k appears in the direct sum as many times as $\sum n_k$. This completes the proof.

EXAMPLE 4.1. A group automorphism on the torus. Let T^n be the n-dimensional torus and A be a group automorphism on T^n. It is known that A can be written in a matrix form. Suppose that A, regarded as a transformation on n-dimensional vector space

Sem. on Probab.
Vol.36 1972年
P1-89

R^n, has a real proper value λ such that $|\lambda| \neq 1$ and has a proper vector r corresponding to λ. Let $\{g_s\}$ be a 1-parameter subgroup of T^n defined by

$$(g_s, g^\wedge) = \exp is\langle r, g \rangle, g \in (T^n)^\wedge,$$

where $(T^n)^\wedge$ is the dual group of T^n and $\langle r, g \rangle$ is the inner product in R^n. Then A and the flow $\{Z_s\}$ which is defined by

$$Z_s g = g + g_s, \quad g \in T^n$$

induce a unitary operator U and 1-parameter group of unitary operators $\{V_s\}$ on $L^2(T^n)$ in a usual manner. It can be easily seen that $\{V_s\}$ is a transversal group for U and that $\{V_s\}$ has the purelyp point spectrum. Consequently, according to the theorem 4.2.1, U has uniform Lebesgue spectrum. Noticing that a unitary operator which has a uniform Lebesgue spectrum is ergodic, we can show that the unitary operator U has exactly σ-Lebesgue spectrum.

Next we shall deal with a unitary operator U which has a transversal group $\{V_s\}$ with continuous spectrum. Let

$$V_s = \int_{-\infty}^{+\infty} e^{2\pi i s \xi} dE^V(\xi), \tag{4.2.7}$$

and

$$H_\xi = E^V(\xi)H. \tag{4.2.8}$$

LEMMA 4.2.3. For any ξ and $m = 0, \pm 1, \pm 2, \ldots$, we get

$$U^m H_\xi = \begin{cases} H_{\xi\lambda^{-m}} & \text{if } \lambda > 0, \text{ or } \lambda < 0 \text{ and } m \text{ is even} \\ H^C_{\xi\lambda^{-m}} & \text{if } \lambda < 0 \text{ and } m \text{ is odd}, \end{cases} \tag{4.2.9}$$

where H_ξ^c means the orthogonal complement of H_ξ .

Proof. Suppose that $\lambda > 0$, or $\lambda < 0$ and m is even. Then we have

(4.2.10) $$(V_s f,g) = \int_{-\infty}^{+\infty} e^{2\pi is\xi}\, d(E^V(\xi)f,g)$$

$$= (U^m V_s f,\ U^m g)$$

$$= (V_{\lambda^m s} U^m f,\ U^m g)$$

$$= \int_{-\infty}^{+\infty} e^{2\pi is\xi}\, d(E^V(\xi\lambda^{-m})U^m f,U^m g) .$$

Clearly $\{ U^{-m}E^V(\xi\lambda^{-m})U^m ,\ -\infty < \xi < +\infty$ is a resolution of the identity and hence the uniqueness of a resolution implies

$$U^m E^V(\xi) = E^V(\xi\lambda^{-m})U^m .$$

Consequently we get

$$U^m H_\xi = H_{\xi\cdot\lambda^{-m}} .$$

Next we suppose that $\lambda < 0$ and m is odd. Put $F(\eta) = I - E^V(\lambda^{-m}\eta-)$, where I is the identity operator. Clearly $\{ F(\eta) \}$ is the resolution of the identity. We get

$$(V_s f,g) = \int_{-\infty}^{+\infty} e^{2\pi is\xi\lambda^m}\, d(E^V(\xi)U^m f,U^m g)$$

$$= \int_{-\infty}^{+\infty} e^{2\pi is\xi\lambda^m}\, d((I - F(\lambda^m\xi))U^m f,U^m g)$$

$$= \int_{+\infty}^{-\infty} e^{2\pi is\eta}\, d((I - F(\eta))U^m f,U^m g)$$

$$= \int_{-\infty}^{+\infty} e^{2\pi is\eta}\, d(F(\eta)U^m f,U^m g) .$$

Sem. on Probab.
Vol.36 1972年
P1-89

From these we get $U^m \cdot E^V(\lambda^{-m}\eta)U^m = U^m E^V(\eta)$ and hence

$$U^m H_\xi = H^c_{\xi\lambda^{-m}} .$$

This completes the proof.

Put $M_m^c = E^V(I_m^{\pm})H$. We can easily see that $\{ U^m M_n$: $m, n=0, \pm 1, \ldots \}$ is an orthogonal family of subspaces, and moreover, from (4.2.9) , we have

$$(4.2.11)\quad U^m M_n = \begin{cases} M_{n+m} & \text{if } \lambda > 0, \text{ or if } \lambda < 0 \text{ and } m \text{ is even} \\ M_{n+m} & \text{if } \lambda < 0 \text{ and } m \text{ is odd.} \end{cases}$$

Let $\{\varphi_k^+\}$ ($\{\varphi_k^-\}$) be a basis of M_0^+ (M_0^-) and let H_k^+ (H_k^-) be the cyclic subspace spanned by $\{U^m\varphi_k^+ : m = 0, \pm 1, \ldots \}$ ($\{U^m\varphi_k^- : m = 0, 1, \ldots\}$) . Then the equalities (4.2.11) imply that for $k = 1, 2, \ldots$

$$\int e^{2\pi i m \xi} \, d\| E^U(\xi)\varphi_k^{\pm}\|^2 = (U^m\varphi_k, \varphi_k) = 0 \quad (m \neq 0)$$

Paley-Wiener's theorem, therefore, implies that U has simple Lebesgue spectrum on each cyclic subspace H_k .

Clearly

$$(4.2.12)\qquad H = \sum \oplus (M_m^+ \oplus M_m^-) = \sum \oplus (U^m M_0^+ \oplus U^m M_0^-)$$
$$= \sum \oplus (H_k^+ \oplus H_k^-) .$$

The subspaces M_0^+ and M_0^- are invariant under the group $\{V_s\}$, and $\{V_s\}$ does not have proper value in M_0^+ and M_0^- . Consequently the dimension of M_0^+ (M_0^-) must be infinite. This means that the components H_k^+ (H_k^-) appear infinitely many times in the decomposition

Sem. on Probab.
Vol.36 1972年
P1-89

(4.2.12).

Summing up the above discussion we get

THEOREM 4.2.2. Aunitary operator U on a separable Hilbert space H has σ-Lebesgue spectrum, if there exists a transversal group $\{V_s\}$ on H with $|\lambda| \neq 1$ the spectrum of which is continuous.

For a unitary operator U, if there exists a transversal group $\{V_s\}$ with $|\lambda| \neq 1$ whose spectrum is not necessarily purely discrete nor purely continuous, we may reduce U and H to those appeared in the previous two particular cases. Let us denote by H_d and H_c the subspaces spanned by all of the proper vectors of $\{V_s\}$ and its orthogonal subspace, respectively. We shall now discuss the case where both H_d and H_c are nontrivial.

THEOREM 4.2.3. If a unitary operator U on a separable Hilbert space H has a transversal group $\{V_s\}$ with $|\lambda| \neq 1$ on H, and if both H_d and H_c are nontrivial, then U has σ-Lebesgue spectrum.

Proof. Let P_d be the projection operator on H_d, and put $P_d V_s = V_s^d$ and $P_d U = U^d$. Clearly P_d commutes with $V_s (-\infty < s < +\infty)$. From Lemma 4.2.1 we get, for any $f \in H_d$, $Uf \in H_d$ and $U^* f \in H_d$. Thus U reduces the subspace H_d. This means that U commutes with P_d, so we have

(4.2.13) $$U^d V_s^d = P_d U P_d V_s = P_d U V_s = P_d^2 V_{\lambda s} U$$

$$P_d V_{\lambda s} P_d U = V^d_{\lambda s} U^d .$$

Thus U^d has a transversal group $\{V^d_s\}$ with the same λ.

Put $(I - P_d)U = U^c$ and $(I - P_d)V_s = V^c_s$. Then $\{V^c_s\}$, likewise the above (4.2.13) constitutes the transversal group of U^c, and according to Theorem 4.2.1 and 4.2.2, we see that U has Lebesgue spectrum on $H = H_d + H_c$, and moreover, at least on H_c, U has σ-Lebesgue spectrum. This completes the proof.

§ 4.3. Spectral type of a 1-parameter group of unitary operators with a transversal group.

Even in case of a 1-parameter group of unitary operators, the analogous discussions to the previous section goes on with a few modifications.

A typical difference between the case of a single unitary operator and that of a 1-parameter group of unitary operators is the fact that only the latter has no transversal group whose spectral measure contains discrete component except point measure concentrating at 0.

THEOREM 4.3.1. The transversal group $\{V_s\}$ of $\{U_t\}$ has purely continuous spectrum on a separable Hilbert space H.

Proof. Suppose there exists = 0 such that

$$V_s \varphi = e^{is\mu}\varphi$$

for all s. Then we get

$$(4.3.1) \qquad (U_{t_1}, U_{t_2}) = (V_s U_{t_1}, V_s U_{t_2}) = \exp(is\, e^{(t_1 - t_2)})(U_{t_1}, U_{t_2}) .$$

This implies that U_{t_1} is orthogonal to U_{t_2}, if $t_1 \neq t_2$.

In other words, $\{ U_t \varphi : -\infty < t < +\infty \}$ is an orthogonal system in H. This contradicts the separablility of H.

For convenience, we shall discuss a group $\{ U_t \}$ which has a transversal group $\{ V_s \}$ with negative λ. Because, if $\lambda > 0$, we can proceed to the same conclusion with slight modification.

We shall omit the proof of the following proposition which is similar to the proof of (4.2.9).

<u>LEMMA 4.3.1.</u> For any ξ and t, we get

$$(4.3.2) \qquad U_t H_\xi = H_{\xi e^{-\lambda t}} .$$

Put $M(t, \xi) = H_{\xi e^{-\lambda t}} \ominus H_\xi$. Then, for suitable $t = t_0$ and $\xi = \xi_0 > 0$, $M^+ = M(t_0, \xi_0)$ is an infinite dimensional subspace, because $e^{-\lambda t}$ is strictly increasing. Let $\{\varphi_n\}$ be a basis of M^+. If $|t| > t_0$,

$$U_t \varphi_1 \in U_t H_{\xi_0 e^{-\lambda t_0}} \ominus U_t H_{\xi_0} \subset H_{\xi_0 e^{-\lambda(t+t_0)}} \ominus H_{\xi_0 e^{-\lambda t_0}} .$$

Consequently we get, if $|t| \geq t_0$,

$$(U_t \varphi_1, \varphi_1) = 0 .$$

Sem. on Probab.
Vol.36 1972年
P1-89

This shows us that $\{U_t\}$ has simple Lebesgue spectrum on the cyclic subspace H_1^+ spanned by $\{U_t\varphi_1, -\infty < t < +\infty\}$. Let P_1 be the projection operator on H_1^+. If $\varphi_2 = P_1\varphi_2$, we proceed to φ_3. If $\varphi_2 \neq P_1\varphi_2$, we shall denote by H_2^+ the cyclic subspace spanned by $\{U_t(\varphi_2 - P_1\varphi_2), -\infty < t < +\infty\}$.

From the equalities

$$(4.3.3) \qquad \begin{aligned} &(U_t(\varphi_2 - P_1\varphi_2), (\varphi_2 - P_1\varphi_2)) \\ &= (U_t\varphi_2, \varphi_2) - (U_tP_1\varphi, P_1\varphi_2) \\ &= -(U_tP_1\varphi_2, P_1\varphi_2), \quad |t| > t_0, \end{aligned}$$

we see that $\{U_t\}$ also has simple Lebesgue specturm on H_2^+. Continuing such a procedure, we obtain a sequence of cyclic subspaces, $H_1^+, H_2^+, \ldots,$ on each of which $\{U_t\}$ has simple Lebesgue spectrum. Noticing that $\varphi_n \in H_1^+ \oplus \cdots \oplus H_n^+$ for any n, we get

$$(4.3.4) \qquad M^+ \subset \sum \oplus H_n^+ \subset H.$$

Starting from $M^- = M(t_1, \xi_1)$ where t_1 and ξ_1 are suitable negative real numbers, the similar construction of the cyclic subspaces $H_n^-(n = 1, 2, \ldots)$ yields

$$(4.3.5) \qquad M^- \subset \sum \oplus H_n^- \subset H.$$

Evidently, $\sum \oplus H_n^+$ is orthogonal to $\sum \oplus H_n^+$. We finally show that H can be expressed as the direct sum of them.

We obtain

Sem. on Probab.
Vol.36 1972年
P1-89

(4.3.6) $$(U_t H_{\xi_0}) \ominus H_0 = U_{t-t_0}(H_{\xi_0 e^{-\lambda t_0}} \ominus H_{\xi_0}) \oplus U_{t-2t_0}(H_{\xi_0 e^{-\lambda t_0}} \ominus H_{\xi_0}) \oplus \cdots$$

$$= U_{t-t_0} M^+ \oplus U_{t-2t_0} M^+ \oplus \cdots ,$$

(4.3.7) $$U_t H_{\xi_1} = U_t(H_{\xi_1 e^{-\lambda t_1}} \ominus H_{\xi_1}) \oplus U_{t+t_1}(H_{\xi_1 e^{-\lambda t_1}} \ominus H_{\xi_1})$$

$$= U_t M^- \oplus U_{t+t_1} M^- \oplus \cdots .$$

Theses imply

(4.3.8) $$U_t H_{\xi_0} \subset (\sum \oplus H_n^+) \oplus H_0$$

and

(4.3.9) $$U_t H_{\xi_1} \subset \sum \oplus H_n^- \subset H_0 .$$

While the continuity of $E^V(\xi)$ and the equation

$$U_t H_{-\xi_0} = H_{-\xi_0 e^{-\lambda t}} = E^V(-\xi_0 e^{-\lambda t})H$$

imply

$$H_0 = E^V(0)H = \sum \oplus H_n^- .$$

Consequently we have

(4.3.10) $$U_t H_{\xi_0} \subset \sum \oplus (H_n^+ \oplus H_n^-) .$$

Now the concluding result

(4.3.11) $$\sum \oplus (H_n^+ \oplus H_n^-) = H$$

can be deduced from the followings

$$U_t H_{\xi_0} = H_{\xi_0 e^{-\lambda t}} = E^V(\xi_0 e^{-\lambda t})H$$

Sem. on Probab.
Vol.36 1972年
P1-89

$$E^V(\xi_0 e^{-\lambda t}) \to I \quad \text{(the identity operator)} \quad \text{as} \quad t \to +\infty \quad .$$

Remark that, contrasting to § 4.2 , we have no information about the number of components in the direct sum $\sum \oplus (H_n^+ \oplus H_n^-)$.

Summing up, we get

THEOREM 4.3.1. A 1-parameter group of unitary operators $\{U_t\}$ on a separable Hilbert space H has uniform Lebesgue spectrum, if there exists a transversal group $\{V_s\}$ with $\lambda \neq 0$ for $\{U_t\}$.

EXAMPLE 4.2. A geodesic flow on a manifold of constant negative curvature. Let us denote the Lie group SL(2,R) by G and denote by D some discrete subgroup of G such that the measure μ on the homogenuous space M = G/D induced from a left invariant measure on G is finite. Then a 1-parameter subgroup $\{g_t\}$ of G induces a group of left transformations on M defined by

$$gD \in M \to g_t gD \in M \ .$$

The dynamical system $(M, \mu, \{g_t\})$ has a realization as a geodesic flow on a manifold of constant negative curvature (refer to [3]) .

Let g be the generator of $\{g_t\}$ and h be a solution of the equation

$$[g, g] = \lambda h$$

for some $\lambda \neq 0$. Put $h_s = e^{sh}$. Then it is straightforward to see that the relation (4.1.2) is satisfied by 1-parameter groups of unitary operators $\{U_t\}$ and $\{V_s\}$ which are induced in a usual manner on $L^2(M)$ by $\{g_t\}$ and $\{h_s\}$ respectively.

Sem. on Probab.
Vol.36 1972年
P1-89

Concequently, from Theorem 4.3.1 , we see that $\{U_t\}$ has uniform Lebesgue spectrum. In [1] Gelfand and Fomin have shown that $\{U_t\}$ has exactly σ-Lebesgue spectrum, appealing to the automorphic function theory.

<u>EXAMPLE 4.3.</u> A flow of the generalized white noise. Let S be Schwartz's space and S^* be its conjugate space. Consider the probability space $P = (S^*, B, \mu)$, where B is the Borel field generated by cylindrical sets in S^* and μ is a white onise. The characteristic functional of μ is given by

$$C_\alpha(\xi) = \exp\left\{-\frac{1}{\alpha}\int |\xi(t)|^\alpha dt\right\} .$$

Let τ_s be the shift operator in S defined by

$$\tau_s \xi(t) = \xi(t-s) , \quad \xi \in S ,$$

and we shall introduce the transformation T_t on S defined by

$$T_t \xi(s) = e^{\frac{\lambda t}{\alpha}} \xi(e^{\lambda t} s) ,$$

where λ is a nonzero real. Then, on the Hilbert space H with the reproducing kernel $C_\alpha(\xi - \eta)$, we get 1-parameter groups of unitary operators $\{U_t\}$ and $\{V_s\}$ induced by $U_t f(\xi) = f(T_t \xi)$ and $V_s f(\xi) = f(\tau_s \xi)$, respectively. They satisfy the relation (4.1.2) , and hence from the Theorem 4.3.1 , we see that $\{U_t\}$ has uniform Lebesgue spectrum.

Sem. on Probab.
Vol.36 1972年
P1-89

V. A measurable flow and the orbit-preserving transformation groups

The main purpose of this section is to study of a measurable flow $\mathcal{T}$ on a standard space with a probability measure, appealing to the group G of bimeasurable transformations which transform the orbits of the flow $\mathcal{T}$ onto other orbits ; we call such a transformation in G an orbit-preserving transformation. Such a group G and its subgroups, as will be shown later, related with many problems in the theory of flow.

The problem that will concern us firstly is that of determining the family of time changed flows of $\mathcal{T}$ which are metrically isomorphic to the flow $\mathcal{T}$; the group G yields such family of time changed flows. The notion of time change of flow was introduced by E. Hopf [7] and is extensively studied by G. Maruyama [14] H. Totoki [22] and R. V. Chacon [28] . Our approach is diffenent from them in the point of view of globa. analysis. Our method is worked by appealing to a cohomologous class of (one) cocycles of the group G ; the notion of additive functionals of a flow which was introduced by G. Maruyama is just cocycle of the group G with respect to the additive group R of real numbers, and a time change function of flow is an inverse function in time variable of an additive functional, although our defänition of time change functions is slightly different from his. These notions are defined in the sections 5.1.

The group G contains important subgroups. As one of them we are concerned with the subgroup G_s in the section 5.4. The group G_s consists of all transformations which transform each orbit of $\mathcal{T}$ onto itself. This group G_s is related with, for example, the time change of an analytic flow defined by a differential equation on the 2-dimensional torus which was treated by I. Arnold [2] amd A. Kolmogorov [9] . In the section 5.5, we extend their results by our method.

The group G also contains a subgroup $\mathcal{A}$. A transformation $\sigma \in G$ belongs to $\mathcal{A}$, if σ is an automorphism and the function $\tau_\sigma = \tau_\sigma(t, \omega)$ defined by

$$\sigma T_t \sigma^{-1} \omega = T_{\tau_\sigma(t, \omega)} \omega , \quad \omega \in \Omega$$

satisfies an admissible condition ;

Sem. on Probab.
Vol.36 1972年
P1-89

$\tau_\sigma = \tau_{\sigma(t,\omega)}$ has a positive derivative at $t = 0$

In terminology of Ya. G. Sinai [21], the flow $\mathcal{T}$ is a transversal flow of σ, which was successfully introduced by him to investigate an automorphism σ (or flow). On the while, our object of study is not σ but the very flow $\mathcal{T}$. Thus, in this sence, our approach is dual to that of Ya. G. Sinai.

The structures of groups $\mathcal{A}$ is related with the value of the entropy of the flow $\mathcal{T}$.
In the section 5.6, we show that if the entropy of $\mathcal{T}$, $h(\mathcal{T})$ is positive finite, the group $\mathcal{A}$ consist of orbit-preserving automorphisms which commute with all T_t, $t \in (-\infty, \infty)$. In the section 5.7, we study the ergodicity and spectrum of $\mathcal{T}$ appealing to the structure of the groups $\mathcal{O}_\omega$ and $\mathcal{A}$.

§ 5.1. Notation and Definition

Throughout this section, $(\Omega, \mathcal{B}, P)$ is a standard space with a probability measure P. Namely $(\Omega, \mathcal{B})$ is a Hausdorff space with a topological Borel field $\mathcal{B}$ such that there exist a completely metrisable topological space Ω' with II-axiom, and f a 1-1, onto continuous mapping of Ω' onto Ω with Borel measurable inverse f^{-1}. Note that if $\mathcal{B}$ is the completion of the topological Borel field of Ω, $(\Omega, \mathcal{B}, P)$ is a Lebesque space,

Two spaces $(\Omega, \mathcal{B}, P)$ and $(\Omega^\circ, \mathcal{B}^\circ, P^\circ)$ are isomorphic if there exists a bimeasurable mapping θ from Ω onto Ω° such that $P^\circ(\theta E) = P(E)$, $E \in \mathcal{B}$. An automorphism on Ω is an 1-1 bimeasurable and measure-preserving transformation on Ω. We mean by a flow an 1-parameter group of automorphisms on Ω .

A flow $\{T_t\}$ is said to be measurable if the following condition is satisfied ;

$$\{(t,\omega)\ ;\ T_t\omega \in B \in \mathcal{B}\} \in \mathcal{B}_R \times \mathcal{B}$$

, where $\mathcal{B}_R$ is the topological Borel field in R , the real line.

Throughout this section, by a flow, we mean a measurable flow and we denote it by $\mathcal{T} = (\Omega, \mathcal{B}, P, T_t)$.

Two flow $\mathcal{T} = (\Omega, \mathcal{B}, P, T_t)$ and $\mathcal{T}^0 = (\Omega^0, \mathcal{B}^0, P^0, T_t^0)$ are said to be (metrically) isomorphic, if there exists an isomorphism θ from Ω onto Ω^0 such that $\theta^{-1}T_t^0\theta\omega = T_t\omega$, a. e. ω (dP) for all $t \in R$, where the abbrebiation a. e. ω (dP) means that an assertion foregoing to it hold for almost every ω with respect to the measure P . A flow $\mathcal{T} = (\Omega, \mathcal{B}, P, T_t)$ is said to be ergodic, if an invariant $\mathcal{B}$-set E , i. e., $T_tE = E$ for all $t \in R$, has measure 0 or 1 .

Periodic point ω of the flow $\mathcal{T}$ is an element of Ω such that $T_t\omega = \omega$ for some nonzero $t \in R$. Throughout this section, we assume that the set of all periodic points is a null set.

It is well known that an ergodic measurable flow $\mathcal{T}$ can be represented by the special flow $\mathcal{T}^* = (\Omega^*, \mathcal{B}^*, P^*, T_t^*)$; the special flow $\mathcal{T}^*$ can be constructed as follows.

Let $(\Omega^0, \mathcal{B}^0)$ be a standared space with measure m , be a positive

Sem. on Probab.
Vol.36 1972年
P1-89

real valued Borel function on Ω° and let T be an automorphism on (Ω°, m). Let μ be the Lebesgue measure on R. Let Ω^* be the set of all pairs of points (p, t) with $0 \leq t \leq \theta(p)$ and $\mathcal{B}^*$ be the restriction of $\mathcal{B}^{\circ} \times \mathcal{B}_R$ to Ω^*. Put $P^* = m \times \mu / N$, where N is a normalizer, and define the 1-parameter group $\{T_t^*\}$ by

$$T_t^*(p, u) = \begin{cases} (T^n p,\ t + u - \sum_0^{n-1} \theta(T^k p)) \\ \quad \text{for } n > 0 \text{ and } \sum_0^{n-1} \theta(T^k p) \leq t + u < \sum_0^n \theta(T^k p) \\ (p,\ t + u) \quad \text{for } 0 \leq u + t < \theta(p) \\ (T^{-n} p,\ t + u + \sum_1^n \theta(T^{-k} p)) \\ \quad \text{for } n > 0 \text{ and } -\sum_1^n \theta(T^{-k} p) < t + u \leq \sum_0^{n-1} \theta(T^{-k} p) \end{cases}$$

Then $\mathcal{J}^* = (\Omega^*, \mathcal{B}^*, P^*, T_t^*)$ is a measurable flow.

Put $F(p, u) = \theta(p)$ and $G(p, u) = u$, and then F and G are measurable function on Ω^*.

It is known that a measurable flow $\mathcal{J} = (\Omega, \mathcal{B}, P, T_t)$ is isomorphic to a special flow $\mathcal{J}^* = (\Omega^*, \mathcal{B}^*, P^*, T_t^*)$. Refer to [17].

§ 5.2. The orbit-preserving transformation group G and time change of flow.

Suppose we are given a measurable flow $\mathcal{J} = (\Omega, \mathcal{B}, P, T_t)$ on a standard space $(\Omega, \mathcal{B})$ with a probability measure P. Concerning this flow $\mathcal{J}$, we wish to introduce a

Sem. on Probab.
Vol.36 1972年
P1-89

group of bimeasurable transformations on $(\Omega, \mathcal{B})$, which make the flow $\mathcal{T}$ invariant in the following sense.

Definition 5.2.1. Let $\mathcal{O}_\omega$ be the orbit of ω under the flow $\mathcal{T}$,

$$\mathcal{O}_\omega = \{T_t\omega \; ; \; -\infty < t < +\infty\} .$$

Let σ be a bimeasurable point transformation on Ω which transforms every orbit onto another orbit ;

$$\sigma(\mathcal{O}_\omega) = \mathcal{O}_\eta \quad , \qquad (5.2.1)$$

We say such mapping σ is orbit-preserving transformation and we denote by G the set of all orbit-preserving transformations on .
Clearly, G forms a group under the ordinary multiplication.

We identify σ_2 with $\sigma_1 \in G$ if the measure of the set $\{\omega \in \Omega \; ; \sigma_1\omega \neq \sigma_2\omega\}$ is zero. This identification is an equivalence relation which is compatible with respect to the group operation.

If no confusion is likely to occur, we use the notation G for the quatient group of G with respect to the above equivalence relation.

The group G contains some subgroups which play important roles in the study of the flow $\mathcal{T}$. We shall give the definitions of them.

Definition 5.2.2. A transformation $\sigma \in G$ is said to be strictly orbit-preserving , if σ transforms every orbit onto itself; $\sigma(\mathcal{O}_\omega) = \mathcal{O}_\omega$. We denote by G_s the set of all strictly orbit-preserving transformations on Ω . The set G_s is a subgroup of G .

By C we mean the set of all commuting transformations with every T_t . The set C also forms a subgroup of G .

Let $\mathfrak{A}$ and $\mathfrak{A}_s$ be the intersection of G and G_s with the set of all automorphisms on Ω , respectively.
They also form subgroups of G . We use sometimes the notion $\mathcal{J}$ for the group $\{T_t\}$.

We can easily see that all the groups introduced above are metrical invariants of the flow $\mathcal{J}$; Suppose that a flow $\mathcal{J}^\circ = (\Omega^\circ, \mathcal{B}^\circ, P^\circ, T_t^\circ)$ is isomorphic to the flow $\mathcal{J}$ with respect to an isomorphism θ .
Then the group G° is isomorphic to the group G with respect to the isomorphism θ , where G is the orbit-preserving group associated with the flow $\mathcal{J}^\circ$. This situation is same to any other groups introduced above.

<u>Proposition 5.2.1.</u> The groups G , G_s, $\mathfrak{A}$, $\mathfrak{A}_s$, and C are metrical invariants of the flow $\mathcal{J}$.

The group G is closely related with the time chage functions of the flow $\mathcal{J}$. The notion of time change function of a flow was introduced by G. Maruyama [14], from which ours is slightly different (refer also to H. Totoki [22]).

<u>Definition 5.2.3.</u> A time change function $\tau = \tau(t,\omega)$ of the flow $\mathcal{J}$ is a real valued function defined on R which satisfies the followings ;

(1) $\tau(t,\omega)$ is a finite valued and 1-1 mapping from R onto itself for a.e.ω.

Sem. on Probab.
Vol.36 1972年
P1-89

(2) $\tau(t+s,\omega) = \tau(t,\omega) + \tau(s, T_{\tau(t,\omega)}\omega)$ for a.e. ω

(3) $\tau(0,\omega) = 0$ a.e. ω.

The set of all time change functions of the flow $\mathcal{T}$ is denoted by $\mathfrak{F}$. When a time chage function $\tau(t,\omega)$ is Borel measurable in (t,ω) we say τ is a measurable time chage function.

Now we shall consider the relation between G and $\mathfrak{F}$. Since any transformation $\sigma \in G$ is orbit-preserving, there corresponds to σ a function $\tau_\sigma(t,\omega)$ by the relation

$$\sigma T_t \sigma^{-1} \omega = T_{\tau_\sigma(t,\omega)}\omega \qquad (5.2.2)$$

To prove the measurability of τ_σ, $\sigma \in G$, we identify the flow $\mathcal{T}$ with the special flow $\mathcal{T}^* = (\Omega^*, \mathcal{B}^*, P^*, T_t^*)$ of $\mathcal{T}$ and we regard the group G and the functions $\tau = \tau(t,\omega)$ as ones associated with the flow $\mathcal{T}^*$. Let Ω_0 be the basic space, θ be the ceiling function and T be the basic automorphism. Let $G(p, x) = x$ and $F(p, x) = \theta(p)$, then G and F are Borel measurable. Let $\sigma \in G$. We define the mappings $(p, t) \to R[(p, t);\sigma]$ and $(p, t) \to L[(p, t);\sigma]$ from Ω^* into Ω_0 and from Ω^* into R, respectively by

$$\sigma(p, t) = (R[(p, t);\sigma], \ L[(p, t);\sigma])$$

When σ is strictly orbit preserving, for each (p, t), there corresponds an integer k such that $R[(p, t);\sigma] = T^k p$. We define

$$K[(p, t);\sigma] = k$$

Sem. on Probab.
Vol.36 1972年
P1-89

Lemma 5.2.1. The functions $R[\ ;\ ]$ and $L[\cdot\,;\sigma]$ are measurable for any $\sigma \in G$, and $K[\cdot\,;\sigma]$ is also measurable for any $\sigma \in G_s$.

Proof. Since the set $\{(p, t);\ R[(p, t);\sigma] \in M\} = \sigma^{-1}(M^*)$ is measurable for any measurable set $M \subset \Omega_0$, $R[\cdot\,;\sigma]$ is measurable. The measurability of $L[\cdot\,;\sigma]$ is deduced form the equation $L[(p, t);\sigma] = G[\sigma(p, t))$. Let $\{\zeta_n\}$ be the sequence of measurable partitions of Ω_0 which satisfies the followings

1) $\zeta_n \uparrow \varepsilon$

2) for any different points $p, q \in \Omega_0$, there exists a partition ζ_n which separates p and q , namely there exists $M \in \zeta_n$ such that $p \in M$ and $q \bar{\in} M$. Let $\sigma \in G_s$

We denote the set $\{(p, t);\ K[(p, t);\sigma] = k\}$ by E_k . Suppose $(p, t) \in E_k$ Then for any n there exists $M_n \in \zeta_n$ such that $T^k p = R[(p, t);\sigma] \in M_n$ and hence $(p, t) \in (T^{-k}M_n)^*$. Thus

$$E_k \subset \bigcap_n \bigcup_{M \in \zeta_n} \{R[(p, t);\sigma] \in M\} \cap (T^{-k}M_n)^*.$$

Conversely, let $(p, t) \in \bigcap \bigcup R[(p, t);\sigma] \in M \cap (T - KM)^*$.
Then there exists a set $M_n \in \zeta_n$ such that $R[(p, t);\sigma] \in M_n$ and $(p, t) \in (T^{-k}M_n)^*$ for any n . Hence $R[(p, t);\sigma] = T^k p$; if not, there exists a partition ζ_n which separates $R[(p, t);\sigma]$ and $T^k p$.

It follows

$$E_k = \bigcap_n \bigcup_{M \in \zeta_n} \{R[(p, t);\sigma] \in M\} \cap (T^{-k}M)^*,$$

Sem. on Probab.
Vol.36 1972年
P1-89

and therfore E_k is the measurable set.

Theorem 5.2.1. 1) $\tau_\sigma = \tau_\sigma(t, \omega)$ is the measurable time change of $\mathcal{J}$ for any $\sigma \in G$.

2) $\tau_{\sigma_1 \sigma_2}(t, \omega) = \tau_{\sigma_1}(\tau_{\sigma_2}(t, \sigma_1^{-1}\omega), \omega)$ a.e. ω (dP).

Proof. Since the flow $\mathcal{J}^*$ is measurable, the mappings $((p, t), s) \to R[(p, t); T_s^*]$ and $((p, t), s) \to L[(p, t); T_s^*]$ are measurable.

Putting $g_k(p, t) = L[(p, t); \sigma^{-1}] + s - \Theta_k(P[(p, t); \sigma^{-1}], L[(p, t); \sigma^{-1}])$,

we get $T_s^* \sigma^{-1}(p, t) =$

$$\sigma((T^k R[(p, t]; \sigma^{-1}], L[(p, t]; \sigma^{-1}] + s - \Theta_k(R[(p, t]; \sigma^{-1}]), L[(p, t); \sigma^{-1}]))$$

$$= (R[(T^k R[(p, t); \sigma^{-1}], g_k(p, t)); \sigma], L[(T^k R[(p, t); \sigma^{-1}], g_k(p, t)); \sigma^{-1}]) ,$$

when $\Theta_k(R[p, t)\sigma^{-1}], L[(p, t); \sigma^{-1}]) \quad L[(p, t); \sigma^{-1} + s$

$$< \Theta_{k+1}(R[(p, t); \sigma^{-1}], L[(p, t); \sigma^{-1}]) .$$

On the while,

$$\sigma T_s^* \sigma^{-1}(p, t) = T^*_{\tau_\sigma(p,t)}(p, t)$$

$$= (T^j p, t + \tau_\sigma(s, (p, t)) - \Theta_j) ,$$

when $\Theta_j \leq t + \tau_\sigma(s, (p, t)) < \Theta_{j+1}$ — (3 . 1 . 2)

It follows

Sem. on Probab.
Vol.36 1972年
P1-89

$$R[(T^k R[(p,\ t);\ \sigma^{-1}],\ g_k(p,\ t));\ \sigma] = T^j p$$

$$L[(T^k R[(p,\ t);\ \sigma^{-1}],\ g_k(p,\ t)),\ \sigma] = t + \tau_\sigma(s,(p,\ t)) - \Theta_j$$

Put $K(p,\ t,\ s) = K[(p,\ t);\ T^*_s]$ and $E_k = ((p,\ t),\ s);\ K(p,\ t,\ s) \in M)$ for a measurable set M. The similar considerations to the above lemma lead us to the equality

$$E_k = \bigcap_n \bigcup_{M \in \mathfrak{S}_n} \{((p,\ t),\ s);\ R[(p,\ t);\ T^*_s] \in M\} \cap (T^{-k}M)^*,$$

and hence $K((p,\ t),\ s)$ is measurable. Put $J((p,\ t),\ s) = j$ if $((p,\ t),\ s)$ satisfies (3.1.2). Then

$$\{ J((p,\ t),\ s) = j,\ K((p,\ t),\ s) = k \}$$
$$= \{K((p,\ t),\ s) = k\} \cap \bigcap_n \bigcup_{M \in \mathfrak{S}_n} \{(R[T^k R[(p,\ t);\ \sigma^{-1}],\ g_k((p,\ t),\ s)] \in M\} \cap (T^{-j}M)^*.$$

and therefore the set $\{J((p,t),s) = j\} = \bigcup_k \{J((p,t),s) = j,\ K((p,t),s)=k\}$ is a measurable set. On the set $\{J((p,t),s) = j\} \cap \{K=k\}$, τ_σ has the form

$$\tau_\sigma(s,(p,t)) = L[T^k R[(p,t);\ \sigma^{-1}],g_k] + \Theta_j(T^k R[(p,t);\ \sigma^{-1} - \tau$$

and hence τ_σ is measurable.

The other properties of τ_σ are easy to see and we omit them.

Thus we have the mapping $:\sigma \to \tau_\sigma$ from G into $\mathfrak{H}$. Moreover the formula (3.1) gives us a time changed flow $s_\sigma = (\Omega, \mathcal{B}_\Omega, Q, S_t)$ of the flow $\mathcal{J}$, where $S_t\omega = T_{\tau_\sigma(t,\omega)}\omega$ and $Q(E) = P(\sigma^{-1}E)$, $E \in \mathcal{B}_\Omega$.

<u>Lemma 5.2.2.</u> $S_\sigma = (\Omega, \mathcal{B}_\Omega, Q,\ S_t)$ is a measurable flow which is metrically isomorphic to the flow $\mathcal{J}$.

Conversely let $\tau = \tau(t,\omega) \in \mathfrak{H}$ be a measurable time change function

Sem. on Probab.
Vol.36 1972年
P1-89

and define an automorphism S_t by $S_t\omega = T_{\tau(t,\omega)}\omega$, $\omega\in\Omega$. Then we can easily see that $\{S_t\}$ is a group of bimeasurable transformations

Suppose that there exists a bimeasurable transformation σ on Ω such that

$$\sigma T_t \sigma^{-1}\omega = S_t\omega, \ \omega\in\Omega .$$

Then, with respect to the probability measure $Q(E) = P(\sigma^{-1}E)$, $E\in\mathcal{B}_\Omega$, $S = (\Omega, \mathcal{B}_\Omega, S_t, Q)$ becomes a measurable flow which is isomorphic to the flow $\mathcal{T}$, with respect to σ. Clearly such transformation σ must be orbit-preserving and $\tau(t,\omega) = \tau_\sigma(t,\omega)$ a.e.ω.(dP). We denote by $\mathcal{F}(G)$ the family of all $\tau_\sigma = \tau_\sigma(t,\omega)$, $\sigma\in G$.

Note that the mapping : $\sigma \to \tau_\sigma$ is not bijection.

We say that two transformations σ_1 and σ_2 $(\in G)$ are <u>equivalent</u> if $\sigma_2\sigma_1^{-1}\in C$ and denote $\sigma_1\sim\sigma_2$. It is trivial to see that the relation $\sim$ is an equivalence relation.

Denote by $G/_\sim$ the quotient space of G with respect to the above relation $\sim$, and by $[\sigma]$ the equivalence class with representative $\sigma\in G$. From the relation $\sigma_1\sim\sigma_2$, it follows $\tau_{\sigma_1} = \tau_{\sigma_2}$ a.e.ω, because of the followings;

$$\begin{aligned} T_{\tau_{\sigma_1}(t,\omega)}\omega &= \sigma_1 T_t \sigma_1^{-1}\omega = \sigma_2\sigma_2^{-1}\sigma_1 T_t \sigma_1^{-1}\omega \\ &= \sigma_2 T_t \sigma_2^{-1}\sigma_1\sigma_1^{-1}\omega = \sigma_2 T_t \sigma_2^{-1}\omega \\ &= T_{\tau_{\sigma_2}(t,\omega)}\omega . \end{aligned}$$

Sem. on Probab.
Vol.36 1972年
P1-89

Thus we see that the mapping : $G/\sim \ni [\sigma] \to \tau_\sigma \in \mathcal{F}(G)$ is a bijection. In the section 5.4, we shall give some results about the characterization of time chage function in $\mathcal{F}(G_s)$.

Summing up the above discussions, we obtain

Proposition 5.2.2. The element $[\sigma] \in G/\sim$ induces the time changed flow $S_{[\sigma]} = (\Omega, \mathcal{B}, Q, S_t)$ which is isomorphic to the flow $\mathcal{T}$ via the time change function $\tau_\sigma \in \mathcal{F}(G)$. Moreover the mapping $G/\sim \ni [\sigma] \to \tau_\sigma \in \mathcal{F}(G)$ is a bijection.

There exists a flow $\mathcal{T}$ such that $\mathcal{F}$ contains a time change function which does not come from the group G . For example, let $\mathcal{T}$ be a flow with discrete spectrum. Then there exists always a time changed flow $\mathcal{S}$ with continuous spectrum [28] so that $\mathcal{S}$ is not isomorphic to $\mathcal{T}$ and the time change function does not belong to $\mathcal{F}(G)$.

5.3. The cohomology $H^1(\mathcal{T}, R)$ and time change functions.

In this section, we discuss time change functions of the flow $\mathcal{T}$ and cocycles of the group $\{T_t\}$. At first, we shall give definitions of cocycles and cohomology of a general dynamical system.

Let (X, B, μ) be a measure space and $\mathcal{G}$ be a transformation group on X such that $\mu(gE) = 0$ if $\mu(E) = 0$, $E \in B$, $g \in \mathcal{G}$. We say that a mapping φ from $\mathcal{G} \times X$ into a group Σ is a cocycle of the dynamical system $(X, B, \mu, \mathcal{G})$ with respect to the group Σ, if φ satisfies the equation

Sem. on Probab.
Vol.36 1972年
P1-89

$$\varphi(g_2 g_1, \omega) = \varphi(g_1, \omega)\varphi(g_2, g_1\omega) \quad \text{a.e. } \omega \text{ and for any } g_1, g_2 \in \mathcal{G}. \tag{5.3.1}$$

We denote by $\tilde{H}^1(\mathcal{G}, \Sigma)$ the set of all cocycles of $\mathcal{G}$.
Two cocycles φ and $\psi \in \tilde{H}^1(\mathcal{G}, \Sigma)$ are said to be <u>homologous</u> with respect to a coboundary $h = h(\omega)$ if there exists a function $h = h(\omega)$ on X with values in Σ such that

$$\varphi(g, \omega)h(\omega) = \psi(g, \omega)h(g\omega) \quad \text{a.e.}\omega. \tag{5.3.2}$$

As can be easily seen, the homologous relation is an equivalence relation. The group of homologous classes of cocycles is called the <u>cohomology</u> of $\mathcal{G}$ with respect to Σ and is denoted by $H^1(\mathcal{G}, \Sigma)$.

Now we consider the dynamical system $\mathcal{T} = (\Omega, \mathcal{B}_\Omega, P, T_t)$ and its cohomology $H^1(\mathcal{T}, R)$. To each time change function $\tau \in \mathcal{H}$, we can construct an additive functional in the following way.
Let $\tau = \tau(t, \omega)$ be a time change function, we define $\varphi = \varphi(u, \omega)$ the inverse function in u of $\tau = \tau(t, \omega)$ by

$$\varphi(u, \omega) = t, \quad \text{if } \tau(t, \omega) = u.$$

By Definition 3.1, φ is well defined on $R \times \Omega$.

<u>Lemma 5.3.1.</u> The functional φ satisfies

a) $\varphi = \varphi(u, \omega)$ is a finite valued 1 - 1 mapping from R onto itself for a.e.ω (dP)

b) $\varphi(u + v, \omega) = \varphi(u, \omega) + \varphi(v, T_u\omega)$ for a.e.ω (dP)

c) $\varphi(0, \omega) = 0$ for a.e.ω (dP).

<u>Proof.</u> It is trivial to see a) and c). We shall show only b). Put $\tau(t, \omega) = u$, $\tau(s, T_{\tau(t,\omega)}\omega) = v$, and $\tau(t + s, \omega) = \gamma$

Sem. on Probab.
Vol.36 1972年
P1-89

Then it follows

$$\varphi(u+v,\omega)=\varphi(\gamma,\omega)=t+s=\varphi(u,\omega)+\varphi(v,T_u\omega).$$

We shall agree to say that the fuctional φ defined above is an additive functional corresponding to a time change function τ (cf. G. Maruyama [14]).

By the above lemma, an additive functional of the flow $\mathcal{J}$ is just a cocycle in $H^1(\mathcal{J}, R)$ corresponding to a time change function, $\tau\in\mathfrak{F}$. The additive functional corresponding to a time change function $\tau_\sigma\in\mathfrak{F}(G)$ enjoys special properties ; one of which is the following lemma and it will be used in the section 5.4.

Lemma 5.3.2. Let $\tau_\sigma\in\mathfrak{F}(G)$ and let $\varphi_{\sigma^{-1}}=\varphi_{\sigma^{-1}}(u,\omega)$ be the additive functional corresponding to $\tau_{\sigma^{-1}}=\tau_{\sigma^{-1}}(u,\omega)$.

Then

$$\tau_\sigma(t,\sigma\omega)=\varphi_{\sigma^{-1}}(t,\omega) \quad \text{a.e.}\,\omega,$$

so that τ_σ is measurable if and only if $\varphi_{\sigma^{-1}}$ is measurable.

Proof. Let $\tau_{\sigma^{-1}}(t,\omega)=u$ and then $\varphi_{\sigma^{-1}}(u,\omega)=t$.

It follows, $\sigma^{-1}T_t\sigma\omega=T_{\varphi_{\sigma^{-1}}(t,\omega)}\omega=T_u\omega$. This implies,

$$\sigma T_u\omega=T_{\varphi_{\sigma^{-1}}(u,\omega)}\sigma\omega=\sigma T_u\sigma^{-1}\sigma\omega=T_{\tau_\sigma(u,\sigma\omega)}\sigma\omega.$$

This gives us the conclusion.

We shall give a sufficient condition for two cocycles φ and ψ are mutially homologous.

Sem. on Probab.
Vol.36 1972年
P1-89

Theorem 5.3.1. Suppose that two measurable cocycles φ and ψ of the flow $\mathcal{T}$ satisfy the condition

$$-\infty < \lim_{S} \frac{1}{S} \int_0^S [\varphi(t,\omega) - \psi(t,\omega)]dt < \infty \quad \text{a.e.}\omega.$$

Then φ and ψ are homologous.

Proof. Put

$$\tilde{h}(\omega) = \lim_{S\to\infty} \frac{1}{S} \int_0^S [\varphi(t,\omega) - \psi(t,\omega)]dt$$

and

$$B = \{\omega; -\infty < \tilde{h}(\omega) < \infty\} .$$

Then it follows for every $\omega \in B$

$$\tilde{h}(T_t\omega) = \lim_{S\to\infty} \frac{1}{S} \int_0^S [\varphi(s,T_t\omega) - \psi(s,T_t\omega)]ds$$

$$= \lim_{S\to\infty} \frac{1}{S} \int_0^S [\varphi(t+s,\omega) - \varphi(t,\omega) - \psi(t+s,\omega) + \psi(t,\omega)]ds$$

$$= \tilde{h}(\omega) - \varphi(t,\omega) + \psi(t,\omega)$$

and $T_tB \subset B$.

Let $h = h(\omega)$ be a measurable extension of $\tilde{h} = h(\omega)$. Then

$$\varphi(t,\omega) + h(T_t\omega) = \psi(t,\omega) + h(\omega) \quad \text{a.e.}\omega.$$

The following is related to the converse problem of the previous theorem.

Theorem 5.3.2. Suppose that a measurable cocycle $\varphi(t,\omega)$ is homologous to $\psi(t,\omega)$:

$$\varphi(t,\omega) + h(T_t\omega) = \psi(t,\omega) + h(\omega) \quad \text{a.e.}\omega.$$

If $h = h(\omega)$ is integrable, we get

$$-\infty < \lim_{S\to\infty} \frac{1}{S} \int_0^S [\varphi(s,\omega) - \psi(s,\omega)]ds < \infty \quad \text{a.e.}\,\omega.$$

Proof. Since $\int_0^S \int_\Omega |h(T_t\omega)|\, d\omega dt = \int_\Omega \int_0^S |h(T_t\omega)|\, dtd\omega < \infty$, $\int_0^S |h(T_t\omega)|dt < \infty$ a.e. ω. It follows

$$\frac{1}{S}\int_0^S [\varphi(\omega) - \psi(\omega)]ds = \frac{1}{S}\int_0^S [h(T_s\omega) - h(\omega)]ds$$

$$= \frac{1}{S}\int_0^S h(T_s\omega)ds - h(\omega) .$$

By the Birkhoff's ergodic theorem, we get

$\lim_{S\to\infty} \frac{1}{S}\int_0^S [\varphi - \psi]ds = \tilde{h}(\omega) - h(\omega)$, $\tilde{h}(\omega)$ is a J-invariant function. This completes the proof.

5.4. A characterization of time change functions $\tau_\sigma \in \mathfrak{F}(G_s)$.

In this section, we shall give genometrical interpretations of the previous discussions in 5.3 .

We shall agree to say that a real valued measurable function f on Ω is admissible if the function $F_\omega(t) = f(T_t\omega) - f(\omega) + t$ is 1-1 mapping on R for a.e. ω.

Let f be an admissible function and define a transformation on Ω by

$$\sigma\omega = T_{f(\omega)}\omega \qquad (5.4.1)$$

Sem. on Probab.
Vol.36 1972年
P1-89

<u>Theorem 5.4.1.</u> Let f be a real valued function on Ω.
Then the transformation σ defined by

$$\sigma\omega = T_{f(\omega)}\omega , \qquad \omega \in \Omega$$

is a strictly orbit preserving transformation, namely $\sigma \in G_s$ if and only f is admissible.

<u>Proof.</u> We shall show that σ is the onto mapping. Let $\xi \in \Omega$ be an arbitary element. Then, for some element $\omega \in \mathcal{O}_\xi$ and some $s \in R$, the element ξ has a form $\xi = T_s \omega$. We can find the time $t \in R$ such that

$$s = f(T_t\omega) + t .$$

It follows

$$\xi = T_{f(T_t\omega)+t}\omega = T_{f(T_t\omega)}T_t\omega = \sigma T_t\omega .$$

To show that σ is 1-1, suppose $\sigma\omega = \sigma\xi$. This implies $T_{f(\omega)}\omega = T_{f(\xi)}\xi$. Hence, by putting $t = f(\omega) - f(\xi)$, we obtain $\xi = T_t\omega$. It follows

$$\sigma\xi = \sigma T_t\omega + T_{f(T_t\omega)+t}\omega = \sigma\omega = T_{f(\omega)}\omega .$$

From this, it follows $f(T_t\omega) - f(\omega) + t = 0$. Since f is admissible, we get $t = 0$, namely, $\xi = \omega$.
To prove the measurability of σ, we identify the flow $\mathcal{T}$ with the specisl represintation $\mathcal{T}^* = (\Omega^*, \mathcal{B}^*, P^*, T_t^*)$ of $\mathcal{T}$ and we regard the group G_s and the admissible function f as ones associated woth $\mathcal{T}^*$. Let Ω_0 be the basic space, θ be the ceiling function and T be the basic automorphism. Let $G(p, x) = x$ and $F(p, x) = \theta(p)$. Then

$G(p, x)$ and $F(p, x)$ are Borel measurable.

Put

$$M^* = \{(p, t) ; p \in M\}, \quad M \text{ is measurable set in } \Omega_0$$

$$M^*(a, b) = \{(p, t) ; a \leqq G(p, t) < b\} \cap M^*$$

and

$$\Theta_k = \begin{cases} \sum_0^{k-1} (T^j p) & k \geqq 1 \\ -\sum_{-1}^{-k} (T^j p) & k \leqq -1 \\ 0 & k = 0 \end{cases}$$

We get

$$\sigma^{-1}(M^*(a, b)) = \bigcup_k \{(p, t) ; (T^k p, k + f(p, t) - \Theta_k(p) \in M^*(a, b) ,$$
$$\Theta_k(p) \leqq t + f(p, t) < \Theta_{k+1}(p)\}$$
$$= \bigcup_k [(T^{-k}M)^* \cap \{(p, t) ; a \leqq G(p, t) + f(p, t) - \Theta_k(p, t) < b\}$$
$$\cap \{(p, t) ; \Theta_k(p, t) \leqq G(p, t) + f(p, t) \leqq \Theta_{k+1}(p, t)\}] .$$

Since the right term is the measurable set, and the family $\{M^*(a, b) ;$ $M \in \mathcal{B}_{\Omega_0}, (a, b) \subset R\}$ generates $\mathcal{B}^*$, σ is measurable. The measurabllity of σ^{-1} is deduced from that of the function g .

Conversely we suppose $\sigma \in G_s$. Recall the notations $R[(p, t) ; \sigma]$, $L[(p, t) ; \sigma]$ and $K[(p, t) ; \sigma]$ defined in the lemma 5.1.1. As shown in the lemma 5.1.1 , the set $E_k = \{(p, t) ; K(p, t)=k\}$ is measurable and $\Omega^* = \bigcup_k E_k$. Since the function f has the form

$$f(p, t) = L[(p, t) ; \sigma] + \Theta_k(p, t) + G(p, t) , \quad (p, t) \in E_k$$

f is measurable,

Sem. on Probab.
Vol.36 1972年
P1-89

This completes the proof.

Let $\sigma \in G_s$. Since $\sigma\omega$ is in θ_ω, we can find a time $t \in R$ such that $\sigma\omega = T_t\omega$. Denote t by $f_\sigma(\omega)$. Moreover, to each $\sigma \in G_s$, as was shown in 3.1, there corresponds a time change function $\tau_\sigma \in \mathcal{F}(G_s)$.

<u>Theorem 5.4.2.</u> A time change function $\tau_\sigma \in \mathcal{F}(G_s)$ has the form ; $\tau_\sigma(t, \omega) = f_\sigma(T_t\omega) - f_\sigma(\omega) + t$, a.e. ω.

We denote by φ_o the ordinary time, namely, $\varphi_o = \varphi_o(t, \omega) = t$ for any $\omega \in \Omega$. Note that φ_o is also a cocycle in $H^1(\mathcal{T}, R)$. We, now, can give a condition for a time change function $\tau = \tau(t, \omega)$ of the flow $\mathcal{T}$ to be induced from a transformation $\sigma \in G_s$.

<u>Theorem 5.4.3.</u> Let $\tau \in \mathcal{F}$ and φ be the additive functional corresponding to τ. Then $\tau = \tau(t, \omega)$ is induced from a transformation $\sigma \in G_s$, if $\varphi = \varphi(u, \omega)$ is homologous to the ordinary time $\varphi_o = \varphi_o(t, \omega) = t$ with respect to an admissible coboundary function f, namely

$$\varphi(t, \omega) = f(T_t\omega) - f(\omega) + t \quad \text{a.e. } \omega .$$

<u>Proof.</u> Suppose $\varphi(t, \omega) = f(T_t\omega) - f(\omega) + t$, where f is admissible. Then f yields a transformation σ in G_s and a time change function τ_σ. By Theorem 5.4.2 and Lemma 5.3.2, we get

$$\varphi(t, \omega) = \tau_\sigma(t, \sigma\omega) = \varphi_{\sigma^{-1}}(t, \omega) \quad \text{a.e. } \omega ,$$

where $\varphi_{\sigma^{-1}}$ is the additive functional corresponding to the time change

Sem. on Probab.
Vol.36 1972年
P1-89

function $\tau_{\sigma^{-1}}$. Hence it follows $\tau(t,\omega) = \tau_{\sigma^{-1}}(t,\omega)$, that is, $\tau \in \mathcal{F}(G_s)$.

Combining the above theorem with Theorem 3.2.2, we get a sufficient condition for $\tau = \tau(t,\omega)$ to be induced from $\sigma \in G_s$.

<u>Corollary 5.4.1.</u> If the additive functional φ of a time change function τ is measurable and satisfies

$$-\infty < \lim_{S\to\infty}\frac{1}{S}\int_0^S [\varphi(t,\omega) - t]dt < +\infty \quad \text{a.e.}\,\omega,$$

then $\tau = \tau(t,\omega)$ is induced from a transformation $\sigma \in G_s$ and moreover the time changed flow of $\mathcal{T}$ by τ is metrically isomorphic to the flow $\mathcal{T}$.

§ 5.5. <u>Time change of an analytic flow on the torus.</u>

As an appliction of the previous discussions, we shall consider the flow which was studied by A. Kolmogorov and I. Arnold, and we shall give an extension of their result.

Let us consider the ergodic flow $\mathcal{T} = (M_2, \mathcal{B}, dxdy, T_t)$ on the 2-dimensional torus M_2 with the normalized Lebesgue measure dxdy, where $\mathcal{B}$ is the topological Borel field and T_t is defined by

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \gamma \quad (\gamma \text{ is an irrational number}).$$

Let $K(x, y)$ be a real valued periodic function on R^2 with period 1 such that

$$0 < K(x,y) \in C^{(k)} \quad (k \geqq 3)$$

and

$$\int_0^1 \int_0^1 K(x, y)dxdy = 1 .$$

Define the additive functional $\varphi = \varphi(t, x, y)$ by

$$\varphi(t, x, y) = \int_0^t K(x+s, y+\gamma s)ds .$$

Then we get the time changed flow $S = (M_2, \mathcal{B}, Q, S_t)$ where S_t is defined by

$$\frac{dx}{dt} = \frac{1}{K(x, y)} \qquad \frac{dy}{dt} = \frac{\gamma}{K(x, y)}$$

and

$$dQ(x, y) = K(x, y)dxdy .$$

Now our problem reads as follows ;

Are two flow $\mathcal{T}$ and S isomorphic?

Suppose that γ satisfies an arithmatic condition such that there exist positive numbers L and H $(G < k-2)$ for which

$$|m + n| > \frac{L}{(|m| + |n|)^H} \tag{5.5.1}$$

holds for any integers m and n .

Let

$$K(x, y) = \sum c_{m,n} e^{2\pi i(mx+ny)}$$

be a Fourier expansion of $K(x, y)$. Then we get

$$\varphi(t,x,y) = t + \sum_{(m,n)\neq(00)} c_{m,n} \frac{e^{2\pi i(m+n\gamma)t} - 1}{2\pi i(m+n)} e^{2\pi i(mx+ny)}$$

and

$$\lim_{S\to\infty}\frac{1}{S} \int_0^S [\varphi - t]dt = -\sum_{(m,n)\neq(0,0)} \frac{c_{m,n}}{2\pi i(m+n\gamma)} e^{2\pi i(mx+ny)} .$$

Sem. on Probab.
Vol.36 1972年
P1-89

we shall show that the right term of the above equation is an absolutely convergent series.

We get the following estimation of the Fourier coefficients $c_{m,n}$.

$$2^{k-1}\pi^k(|m| + |n|)^k \; |c_{m,n}| \leq \text{Max}\left\{ \text{Max}\left|\frac{\partial^k}{\partial x^k} K\right|, \text{Max}\left|\frac{\partial^k}{\partial y^k} K\right|\right\}$$

Denoting the right term by M , we get

$$\frac{|c_{m,n}|}{2\pi|m+n|} \leq M/2^k\pi^{k+1}L(|m| + |n|)^{k-H}$$

Let $N(j)$ be a number of the lattice points (m,n) for which $|m| + |n| = j$.

Then

$$N(j) \leqq 2^2(j+1) \leq 2^3 j \; .$$

It follows

$$\sum_{(m,n)\neq(0,0)} \frac{|c_{m,n}|}{2\pi|m+n|} \leq \frac{M}{2^k\pi^{k+1} L} \sum_{(m,n)\neq(0,0)} (|m| + |n|)^{-k+H} ,$$

$$\leq \frac{M}{2^k\pi^{k+1}2^{-3}L} \sum_{j=1} j^{H-k+1}$$

Since $0 < H < k-2$, the right term converges. By Corollary 3.3.1, we conclude that the two flow $\mathcal{T}$ and S are isomorphic with respect to a strictly orbit-preserving transformation.

<u>Theorem 5.5.1.</u> Let $\mathcal{T}$ be the flow on the 2-dimensional torus M_2 defined by

$$dx/dt = 1 \qquad dy/dt = \gamma$$

with the Lebesgue measure $dxdy$, and let S be the time changed flow of $\mathcal{T}$ defined by

$$dx/dt = 1/K(x, y) \qquad dy/dt = \gamma/ K(x, y)$$

with the measure $dQ(x, y) = K(x, y)dxdy$. Suppose $0 < K \in C^{(k)}$ and $\iint K dxdy = 1$, and let γ be an irrational number for which there exist positive numbers L and $H (H < k-2)$ such that

$$\left| m + n\gamma \right| > \frac{L}{(|m| + |n|)^H} \quad \text{for any integers } m \text{ and } n .$$

Then the time changed flow S is isomorphic to the flow $\mathcal{J}$.

5.6. The dynamical system $(\Omega, \mathcal{B}, P, \mathcal{A})$ and the entropy of the flow $\mathcal{J}$.

Let $\mathfrak{F}(\mathcal{A})$ be the set of all time change functions $\tau_\sigma = \tau_\sigma(t, \omega)$, $\sigma \in \Omega$, each of which has the positive derivative at $t = 0$ for a.e. ω. Let

$$\mathcal{A} = \{ \sigma \in \mathfrak{N} ; \tau_\sigma \in \mathfrak{F}(\mathcal{A}) \}.$$

If $\sigma_1, \sigma_2 \in \mathcal{A}$, we get

$$\lim_{t \to 0} \frac{\tau_{\sigma_1 \sigma_2}(t, \omega)}{t} = \lim_{\tau_{\sigma_2} \to 0} \frac{\tau_{\sigma_1}(\tau_{\sigma_2}(t, \sigma_1^{-1}\omega)\omega)}{\tau_{\sigma_2}(t, \sigma_1^{-1}\omega)} \lim_{t \to 0} \frac{\tau_{\sigma_2}(t, \sigma_1^{-1}\omega)}{t}$$

Since $t = \tau_{\sigma^{-1}}(\tau_\sigma(t, \sigma\omega), \omega)$,

$$\lim_{s \to 0} \frac{\tau_{\sigma^{-1}}(s, \omega)}{s} = \lim_{t \to 0} \frac{t}{\tau_\sigma(t, \sigma\omega)} \qquad \text{a.e. } \omega ,$$

So that the set $\mathcal{A}$ forms a subgroup of $\mathfrak{N}$.

Since $\{T_t\} \subset \mathcal{A}$ the dynamical system $\mathcal{A} = (\Omega, \mathcal{B}, P, \mathcal{A})$ is an extension of the dynamical system $\mathcal{J} = (\Omega, \mathcal{B}, P, T_t)$.

We shall show the structure of $\mathcal{A}$ is related to the entropy of the flow $\mathcal{J}$. In the following, we assume the flow is ergodic.

Sem. on Probab.
Vol.36 1972年
P1-89

Lemma 5.6.1. (Ya. G. Sinai [21]) Let $\mathcal{T}$ be an ergodic flow and let τ_σ be a measurable time change function of $\mathcal{T}$ with $\sigma \in \mathcal{A}$. Then we get

$$\tau_\sigma(t, \omega) = \lambda_\sigma t \quad \text{a.e. } \omega, \text{ where } \lambda_\sigma > 0 .$$

Using the above lemma we get

Theorem 5.6.1. Suppose the entropy of the ergodic flow $\mathcal{T}$ is po positive finite. Then

$$\mathcal{A} = \mathfrak{N} \cap C .$$

Proof. Let $S = (\Omega, \mathcal{B}, P, S_t)$ be a time changed flow of the flow $\mathcal{T}$ by τ_σ, $\sigma \in \mathcal{A}$. Since S_t is P-measure invariant, we get

$$h(T_1) = h(S_1) = h(T_\lambda) = \lambda\, h(T_1) .$$

This implies $\lambda = 1$. Hence it follows

$$\sigma T_t \sigma^{-1} \omega = S_t \omega = T_t \omega$$

namely, $\sigma \in C$. Clearly $\mathcal{A} \supset \mathfrak{N} \cap C$. This completes the proof.

The following is a restatement of the above result.

Corollary 5.6.1. Suppose that the flow $\mathcal{T}$ is ergodic. If there exists an automorphism $\sigma \in \mathcal{A}$ which does not commute with some $T_{t_0} \in \{T_t\}$, then the entropy $h(\mathcal{T})$ of the flow $\mathcal{T}$ is zero or infinite.

It seems to me very interesting to show the converse assertion of the theorem 5.6.1, but it is still open. As to the group $\mathcal{A} \cap G_s$, we

Sem. on Probab.
Vol.36 1972年
P1-89

get

Theorem 5.6.2. Let $\mathcal{T}$ be an ergodic flow. Then

$$\{T_t\} = G_s \cap \mathcal{O} \cap \mathcal{C} .$$

Proof. Let $G_s \cap \mathcal{O} \cap \mathcal{C}$. Then $\lambda_\sigma = 1$ and $\tau_\sigma(t, \omega) = f_\sigma(T_t\omega) - f_\sigma(\omega) + t = t$, a.e. ω. Since $\mathcal{T}$ is ergodic, f_σ must be a constant, say $f_\sigma(\omega) = c$ a.e. ω. This implies $\sigma_\omega = T_c\omega$ a.e. ω.

There exist, in fact, ergodic flows such that $\mathcal{A}$ is not included in C .

Example 1. Let A be an ergodic continuous group automorphism on the 2-dimensional torus M_2 . Then the eigenvalues of A are the irrational algebraic numbers of degree 2 ; we identify the group automophism A with the unimodular integral matrix associated with it. We denote one of them by λ and let $(1, \gamma)$ be the eigenvector of A with respect to λ . Put $g_t = (t, \gamma t) \pmod 1$. Then the family $\{g_t\}$ is the 1-parameter subgroup of M_2 , and then the flow defined by

$$T_t g = g + g_t \qquad g \in M_2 ,$$

is ergodic and $AT_tA^{-1} = T_{\lambda t}$. Thus $A \in \mathcal{A}$ and A does not commute with the flow.

Remark. Until now we have discussed a flow with continuous time parameter. The notions of orbit-preserving transformation groups and other concepts introduced previously are avairable also to a flow with discrete parameter, i.e. , an automorphism.

Sem. on Probab.
Vol.36 1972年
P1-89

In this remark, concerning to Theorem 4.1.1, we wish to mention to Bernoulli shift. This is an example of the dynamical system with positive finite entropy for which $\{T_t\} = \mathfrak{O} \cap \mathcal{C}$.

Let $X = \{ x_1, x_2, \ldots, x_n \}$ be a set with a measure

$$p_k > 0 \quad \text{and} \quad \sum p_k = 1 .$$

Let $\mathcal{J} = (\Omega, \mathcal{B}, P, T)$ be a two sided Bernoulli shift defined in an usual manner, where

$$\Omega = \prod_{-\infty}^{\infty} \otimes X_i \quad , \quad X_i = X$$

and

$$(T\omega)_i = \omega_{i+1} \; , \quad \omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \quad L$$

Let L be a permutation of X, and define a transformation σ_L by

$$\sigma_L(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) = (\ldots, L\omega_{-1}, L\omega_0, L\omega_1, \ldots)$$

For convenience, we call σ_L a permutation of Ω.

G.A. Hedlund, M.L. Curtis, and R.C. Lyndon have determined a class of the continuous transformations commuting with shift on a symbolic dynamical system [6]. By using their results, we can determine the group $\mathcal{A} \cap K$ associated with the Bernoulli shift, where K is the group of all continuous transformations on Ω. We define the equivalence relation $\sim$ by

$$x_i \sim x_j \; , \quad \text{if} \quad p_i = p_j .$$

Let $\{X_1, X_2, \ldots, X_k\}$ be the partition of X into the equivalent classes, and let $L(X, P)$ be the set of all permutations which preserve

Sem. on Probab.
Vol.36 1972年
P1-89

each X_j .

Theorem 5.6.3 The group $\mathcal{A} \cap \mathcal{K}$ is generated by T and $\{\sigma_L ; L \in L(X, P)\}$. Moreover, if p_j's are all different, the group $\mathcal{A} \cap \mathcal{K}$ coincides with the group $\mathcal{T} = \{T^m\}$.

§ 5.7. Ergodicity and spectrum of the flow $\mathcal{T}$ and the groups $\mathfrak{A}_s$ and $\mathcal{A}$

We shall give a condition of ergodicity of the flow $\mathcal{T}$ appealing to $\mathfrak{A}_s$.

Theorem 5.7.1. The dynamical system $\mathfrak{A}_s = (\Omega, \mathcal{B}, P, \mathfrak{A}_s)$ is ergodic.

Proof. Since $\{T_t\} \subset \mathfrak{A}_s$, 'if part' is trivial to see. Suppose $\mathfrak{A}_s$ is ergodic. Let E be a $\mathcal{B}$-set which is invariant under $\{T_t\}$. It is enough to show that E is invariant under any $\sigma \in \mathfrak{A}_s$. Let $\omega \in E$. Then $\sigma\omega = T_{f_{\mathcal{T}}}(\sigma, \omega)\omega \in E$; namely, $\sigma E \subset E$. Put $\eta = T_{-f_{\mathcal{T}}}(\sigma^{-1}\xi)\xi$. Then $\eta \in E$ and moreover $\xi = \sigma\sigma^{-1}\xi = \sigma T_{-f_{\mathcal{T}}(\sigma^{-1}\xi)}\xi = \sigma\eta$. This shows $\sigma E = E$.

From the above theorem, we get the following criterion for the flow to be ergodic.

Corollary. If $\mathfrak{A}_s$ contains an ergodic element, then the flow is ergodic.

Now we shall give a characterization of the spectrum of the flow $\mathcal{T}$ appealing to the structure of the group $\mathcal{A}$.

Sem. on Probab.
Vol.36 1972年
P1-89

Theorem 5.7.2. Suppose that the flow $\mathcal{J}$ is ergodic. If $\mathcal{A}$ contains an 1-parameter subgroup $\{\sigma_s ; s \in R\}$ such that $\sigma_s \overline{\in} C$ for any $s \in R$. Then the flow $\mathcal{J}$ is weakly mixing.

Proof. Let $\{U_t\}$ and $\{V_t\}$ be the unitary operators induced from $\{\sigma_t\}$ and $\{T_t\}$; $U_t F(\omega) = F(\sigma_t \omega)$ and $V_t F(\omega) = F(T_t \omega)$, $F \in L^2(\Omega, P)$. Suppose that the flow $\mathcal{J}$ has an eigenvalue $\mu \neq 0$ and eigenfunction F_μ ; $V_t F_\mu = \exp(2\pi i \mu t) F_\mu$, $t \in R$. Since $\sigma_s \overline{\in} C$ and $\sigma_s \in \mathcal{A}$, there exists a function $\lambda(s)$ such that $\sigma_s T_t \sigma_s^{-1} = T_{\lambda(s)t}$ by Lemma 4.1.1. Then we see that for any t, s,
$\lambda(s) \neq 1$, $\lambda(s)\lambda(t) = \lambda(s+t)$ and
$\tau_{\sigma_s}(t, \omega) = \lambda(s)t$. It follows

$$V_t U_s F_\mu(\omega) = F_\mu(\sigma_s T_t \omega) = F_\mu(T_{\lambda(s)t}\sigma_s \omega)$$

$$= U_s V_{\lambda(s)t} F_\mu(\omega) = \exp(2\pi i \lambda(s) \mu t) U_s F_\mu(\omega) .$$

Thus $\{U_s F_\mu ; -\infty < s < \infty\}$ is a family of eigenfunctions corresponding to the eigenvalues $\lambda(s)\mu$. This contradicts to the separability of $L^2(\Omega, P)$.

Appealing to the above theorem, we can see the horocycle flow on the manifold with a constant negative curvature and the flow induced from the Brownian motion are weakly mixing [10] .

Sem. on Probab.
Vol.36 1972年
P1-89

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Vol.36 1972年
P1-89

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Sem. on Probab.
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