VII. FOURIER ANALYSIS OF SAMPLE FUNCTIONS OF STATIONARY PROCESSES

1. Generalisation of Fourier Transforms.

(a) Formal Definitions.

Fourier Transform; $f(t) = \mathfrak{F}g(t) = \int e^{i\lambda t}\, g(\lambda)\, d\lambda$

Inverse Fourier transform; $g(\lambda) = \bar{\mathfrak{F}}f(\lambda) = \frac{1}{2\pi}\int e^{-i\lambda t}\, f(t)\, d\lambda$

(b) $g \in L^2 \rightleftarrows f \in L^2$

$$f(t) = \mathfrak{F}g(t) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} e^{i\lambda t}\, g(\lambda)\, d\lambda$$

$$\Updownarrow$$

$$g(\lambda) = \bar{\mathfrak{F}}f(\lambda) = \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-A}^{A} e^{-i\lambda t}\, f(t)\, dt$$

(c) $g \in L^2 \underset{\not\leftarrow}{\rightarrow} f$ continuous, vanishing at $\pm\infty$

$$f(t) = \mathfrak{F}g(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, g(\lambda)\, d\lambda \qquad \text{(Lebesgue integral)}$$

$$\Downarrow$$

$$g(\lambda) = \bar{\mathfrak{F}}(t) = \text{a.e.} \lim_{\varepsilon\to 0}\ \lim_{A\to\infty} \frac{1}{2\pi}\int_{-A}^{A} e^{-i\lambda t}\, \frac{e^{-it\varepsilon}-1}{-it\varepsilon}\, f(t)\, dt$$

(d) dG complex measure of bounded variation

$\rightleftarrows$ $f = \mathfrak{F}\, dG$ continuous, bounded

$$\mathfrak{F}(dG)(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, dG(\lambda) \qquad \text{(Stieltjes integral)}$$

$$G(b) - G(a) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{A} \int_{a}^{b} e^{-it\lambda}\, d\lambda\, f(t)\, d\lambda \qquad \text{(P. Lévy's formula)}$$

$$G(x) = \frac{G(x+0) + G(x-0)}{2}$$

Special Case. dG purely discontinuous

$\rightleftarrows$ $f = \mathfrak{F}\, dG$ almost periodic function in Bohr's sense

(e) S. Bochner: integrated Fourier transform

(f) L. Schwartz: Fourier transform of slowly increasing distributions.

2. Fourier Transform of Slowly Increasing Distributions.

(a) Space $\mathcal{S}$: $f \in \mathcal{S} \underset{\text{def}}{\longleftrightarrow} f \in C^{\infty}$ and $|x^m f^{(n)}(x)|$ bounded for each (m,n)

$$\|f\| = \sum_{m,n} 2^{-(m+n)} \frac{\|f\|_{m,n}}{1 + \|f\|_{m,n}}$$

$$\|f\|_{m,n} = \sup_x |x^m f^{(n)}(x)|$$

$$\|f_p\| \to 0 \qquad (p \to \infty)$$

$$\rightleftarrows\ x^m f_p^{(n)}(x) \to 0 \qquad \text{uniformly in } x \quad (p \to \infty)$$

$f \in \mathcal{S}$ is called a <u>rapidly decreasing function</u>.

$\mathcal{S}$ is a linear topological space (in fact a Frechet space) invariant under derivation and Fourier transform.

(b) <u>Space</u> $\mathcal{S}'$: $\mathcal{S}'$ is the dual space of $\mathcal{S}$, i.e.,

$$u \in \mathcal{S}' \underset{\text{def}}{\rightleftarrows} u \text{ is a continuous linear functional defined on } \mathcal{S}$$

$\langle u, \varphi \rangle$ = the value of $u(\in \mathcal{S}')$ at $\varphi(\in \mathcal{S})$.

$u \in \mathcal{S}'$ is called a <u>slowly increasing distribution</u>.

$\mathcal{S}'$ is a topological space invariant under derivation and Fourier transform

(i) <u>topology in</u> $\mathcal{S}'$. pseudo-topology $u_n \to 0 \iff \langle u_n, \varphi \rangle \to 0$ for all φ.

(ii) <u>derivation in</u> $\mathcal{S}'$ $\langle Du, \varphi \rangle \underset{\text{def}}{=} - \langle u, \varphi' \rangle$

(iii) <u>Fourier transforms in</u> $\mathcal{S}'$

$$\langle \mathfrak{F} u, \varphi \rangle \underset{\text{def}}{=} \langle u, \mathfrak{F} \varphi \rangle$$

$$\langle \overline{\mathfrak{F}} u, \varphi \rangle \underset{\text{def}}{=} \langle u, \overline{\mathfrak{F}} \varphi \rangle$$

It is easy to see $\overline{\mathfrak{F}} = \mathfrak{F}^{-1}$, $\mathfrak{F} = \overline{\mathfrak{F}}^{-1}$

Example 1. A function g is called a slowly increasing function if

$$\int_{-\infty}^{\infty} \frac{|g(\lambda)|}{1 + |\lambda|^p} \, d\lambda < \infty \qquad \text{for some } p > 0$$

Such function g is a slowly increasing distribution in the sense

$$\langle g, \varphi \rangle = \int_{-\infty}^{\infty} g(\lambda) \, \varphi(\lambda) \, d\lambda$$

It is easy to see

$$\mathcal{F}g = \lim_{A \to \infty} (\text{in } \mathcal{S}') \int_{-A}^{A} e^{it\lambda} \, g(\lambda) \, d\lambda$$

Example 2. A locally bounded variation complex measure dG is called a slowly increasing measure if

$$\int_{-\infty}^{\infty} \frac{|dG|(\lambda)}{1 + |\lambda|^p} \, d\lambda < \infty \qquad \text{for some } p > 0$$

Such measure is a slowly increasing distribution in the sense

$$\langle dG, \varphi \rangle = \int_{-\infty}^{\infty} \varphi(\lambda) \, dG(\lambda)$$

It is easy to see

$$\mathfrak{F}dG = \lim_{A \to \infty} (\text{in } \mathcal{A}) \int_{-A}^{A} e^{it\lambda}\, dG(\lambda)$$

Example 3.

$$f \in L^p \ (1 \leq p \leq \infty)$$

$$\Longrightarrow f \text{ is a slowly increasing function}$$

Example 4. δ is clearly a slowly increasing measure and

$$\mathfrak{F}\delta = 1, \qquad \overline{\mathfrak{F}}\delta = \frac{1}{2\pi}$$

$$\mathfrak{F}1 = \overline{\mathfrak{F}}^{-1} 1 = 2\pi\delta$$

$$\overline{\mathfrak{F}}1 = \mathfrak{F}^{-1} 1 = \delta$$

Example 5.

$$\begin{aligned}
\langle \mathfrak{F} D\delta, \varphi \rangle &= \langle D\delta, \mathfrak{F}\varphi \rangle \\
&= \langle \delta, D\mathfrak{F}\varphi \rangle \\
&= D\mathfrak{F}\varphi(0) \\
&= \frac{d}{d\lambda} \int_{-\infty}^{\infty} e^{i\lambda t} \varphi(t)\, dt \Big|_{\lambda=0} \\
&= \int_{-\infty}^{\infty} e^{i\lambda t}\, it\varphi(t)\, dt \Big|_{\lambda=0} \\
&= \int_{-\infty}^{\infty} it\varphi(t)\, dt
\end{aligned}$$

$\therefore\ \mathfrak{F}(D\delta) = it$

Example 6. If dG is a slowly increasing measure, then $G(\lambda)$ is a slowly increasing function and

$$DG = dG.$$

3. Spectral Decomposition of the Sample Functions of Weakly Stationary Processes.

(a) Let $x(t) = x(t, \omega)$ be a weakly stationary process with

$$(1) \qquad (x(t), 1) = 0$$

and

$$(2) \qquad r(t) \equiv (x(t+s), x(s))$$

is continuous.

Then we have

$$(3) \qquad \text{Hinčin decomposition: } r(t) = \int e^{it\lambda}\, dF(\lambda)$$

$$(4) \qquad \text{Kolmogorov-Cramer decomposition: } x(t) = \int e^{it\lambda}\, dM(\lambda)$$

(b) Regularization of $x(t)$.

Theorem 1. Given a weakly stationary process with (1) and (2), there exists a function $f(t, \omega)$ Borel measurable in (t, ω) such that, for each t,

$$(5) \qquad f(t, \omega) = x(t, \omega) \qquad \text{for a.a. } \omega$$

Such $f(t, \omega)$ is uniquely determined up to $dt \cdot dP(\omega)$-measure 0 and is called the regularization of $x(t)$.

Theorem 2. The regularization $f(t, \omega)$ in Theorem 1 is a slowly increasing function of t for a.a. ω. In fact

(6) $$\int_{-\infty}^{\infty} \frac{|f(t, \omega)|}{1 + t^2} dt < \infty, \qquad \text{for a.a } \omega$$

(c) Regularization of $M(\lambda)$. Let $M(\lambda)$ denote $M(-\infty, \lambda]$. Then $M(\lambda)$ is right continuous in the $L^2(\Omega)$-norm and has orthogonal increments.

Theorem 3. There exists a function $G(\lambda, \omega)$ Borel measurable in (λ, ω) such that, for each λ,

(6) $$G(\lambda, \omega) = M(\lambda, \omega), \qquad \text{for a.a. } \omega$$

Such G is uniquely determined up to $d\lambda \cdot dP(\omega)$ - measure 0 and is called the regularization of $M(\lambda, \omega)$

Theorem 4. The regularization $G(\lambda, \omega)$ is a slowly increasing function of λ for a.a. ω and so $D_\lambda G \in \mathcal{S}'$ for a.a ω. In fact

$$\int \frac{|G(\lambda, \omega)|}{1 + \lambda^2} d\lambda < \infty \qquad \text{for a.a. } \omega$$

(d) Spectral Decomposition. Let $x(t)$ be a weakly stationary process with the Kolmogorov-Cramer decomposition

$$x(t) = \int e^{i\lambda t} dM(\lambda)$$

and $f(t, \omega)$ and $G(\lambda, \omega)$ be the regularizations of $x(t)$ and $M(\lambda)$ respectively.

Theorem 5. $f = \mathcal{F}[DG]$ for a.a. ω where $\mathcal{F}$ is the Fourier transform in $\mathcal{S}'$.

(e) Examples. We use the same notations as before. By taking Doob's separable version if necessary, we can assume that $f(t,\omega)$ and $G(\lambda,\omega)$ are separable and measurable.

Ex.1. If $r(t) = \sum e_n e^{i\lambda_n t}$ with $\sum_n \sqrt{e_n} < \infty$, then

$$f(t,\omega) = \sum_n M_n(\omega)\, e^{i\lambda_n t},$$

where $M_n = M(\lambda_n+0) - M(\lambda_n-0)$, $\sum_n |M_n| < \infty$ a.e.

$$G(\lambda,\omega) = \sum_{\lambda_n<\lambda} M_n \quad \text{bounded variation on } (-\infty,\infty).$$

Ex.2. If $x(t)$ is Gaussian and $F(\lambda)$ is continuous and strictly increasing, then

$f(t,\omega)$ is unbounded in $-\infty<t<\infty$ for a.a. ω.

$G(\lambda,\omega)$ is continuous in $-\infty<\lambda<\infty$ but not of bounded variation on any small λ-interval for a.a. ω.

Ex.3. If $x(t)$ is Gaussian, then $G(\lambda,\omega)$ has only the first kind discontinuities in λ for a.a. ω.

Ex.4.(Yu. K. Belajev) If $r(t)$ is analytic in $-\infty<t<\infty$ (for example if $\int_{-\infty}^{\infty} e^{c|\lambda|}\, dF(\lambda) < \infty$ for every $c>0$, in particular if the support of dF is compact), then $f(t,\omega)$ is analytic in $-\infty<t<\infty$ for a.a. ω.

Ex.5. There exists a strictly stationary process whose sample paths (regularization) are unbounded on every interval. For example, take a periodic function $g(t)$ with period 1 which is square summable on $[0,1]$ and unbounded on every interval; the existence of such g is easily seen. Now set

$$x(t,\omega) = g(t+\omega), \quad \omega \in \Omega \equiv [0,1].$$

Then $x(t,\omega)$ satisfies our conditions.

4. Generalized Harmonic Analysis of Sample Functions of Strictly Stationary Processes.

Definition 1. A complex valued function $f(t)$ of a real variable $t = (-\infty, \infty)$ is said to belong to the Wiener class (or Wiener-Hopf class) $\mathcal{W}$ if

(1) $f(t)$ is measurable

(2) $R(t) \equiv R(t : f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t+s)\overline{f(s)}\, ds$ is convergent for a.a. t including $t = 0$.

(3) $\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} R(t)\, dt = R(0)$

Remark 1. Wiener required the existence of $R(t)$ at every point t and its continuity. Hopf defined $R(t)$ by

(2') $$\lim_{\substack{A \to -\infty \\ B \to \infty}} \frac{1}{B-A} \int_{A}^{B} f(t+s)\overline{f(s)}\, ds$$

instead of (2).

Remark 2. We can deduce the following properties from (1) and (2) without using (3)

(4) $$|R(t)| \leq R(0) \qquad \text{a.e.}$$

(5) $\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} R(t)\,dt$ is real and $\leq R(0)$

(6) $R(t)$ is a measurable positive definite function in the sense that $\int\int R(t-s)\,\varphi(t)\,\overline{\varphi(s)}\,dt\,ds \geq 0$ for every continuous function φ with compact support and so it has Bochner's representation

$$R(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\,d\Psi(\lambda) \qquad \text{for a.a.t.}$$

where $d\Psi$ is a bounded measure on $(-\infty, \infty)$.

Thus the limit of the left side in (3) exists and is $\leq R(0)$. The condition (3) claims that the equality holds.

Remark 3. (3) does not follow from (1) and (2). See the example $f(t) = \sin(t^2)$. Then $R(t) \equiv 0$ for $t \neq 0$ and $R(0) = 1/2$.

Definition 2. $R(t)$ is called the auto-correlation function of $f(t)$ and $d\Psi$ is called the spectral measure of f.

Example 1. $f(t) = \Sigma\, a_n e^{i\lambda_n t}$

$\Longrightarrow f \in \mathcal{W}$ and $R(t) = \Sigma\, |a_n|^2 e^{i\lambda_n t}$

Example 2. $f(t) = \int e^{i\lambda t}\,dG(\lambda) \qquad \int |dG(\lambda)| < \infty$

$\Longrightarrow f \in \mathcal{W}$ and $R(t) = \Sigma\, e^{i\lambda_n t} |a_n|^2$

where $\{\lambda_n\}$ are points of discontinuity of G and c_n is the jump at a_n for each n.

In these examples the spectral measures are purely discontinuous. See Theorem 4 for the existence of a function $\in \mathcal{W}$ with given general spectral measure.

Theorem 1. If $f \in \mathcal{W}$, then

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2}\, dt < \infty \tag{7}$$

and f is therefore a slowly increasing function.

Proof. Set $S_n = \frac{1}{n}\int_{-n}^{n} |f(s)|^2\, ds$. Then S_n is convergent by (2) and so bounded

$$\begin{aligned}
\int_{-N}^{N} \frac{|f(t)|^2}{1+t^2}\, dt &= \sum_{n=0}^{N-1} \int_{n \le |t| < n+1} \frac{|f(t)|^2}{1+t^2}\, dt \\
&\le \sum_{n=0}^{N-1} \frac{1}{1+n^2}\left[(n+1)\, S_{n+1} - n\, S_n\right] \qquad (S_0 = 0) \\
&= \sum_{n=1}^{N} \left[\frac{1}{1+(n-1)^2} - \frac{1}{1+n^2}\right] n\, S_n \\
&= \sum_{n=1}^{N} O(1/n^2)
\end{aligned}$$

which implies (7). f is then a slowly increasing function by virtue of Schwarz inequality.

By Theorem 1 we can express f as the Fourier transform of a distribution $\in \mathcal{S}'$, but we have a more concrete expression.

Theorem 2. (N. Wiener). If $f \in \mathcal{W}$, then

$$(8) \qquad g(\lambda) = \frac{1}{2\pi}\int_{-1}^{1} f(t)\,\frac{e^{-i\lambda t}-1}{-it}\,dt + \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\left[\int_{1}^{A}+\int_{-A}^{-1}\right] f(t)\,\frac{e^{-i\lambda t}}{-it}\,dt$$

is well defined and we have

$$(9) \qquad f(t) = \lim_{\epsilon\downarrow 0}\ \underset{A\to\infty}{\text{l.i.m.}}\int_{-A}^{A} e^{i\lambda t}\,\frac{g(\lambda+\epsilon)-g(\lambda-\epsilon)}{2\epsilon}\,d\lambda$$

$$(10) \qquad R(t:f) = \lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty} e^{i\lambda t}\,\frac{|g(\lambda+\epsilon)-g(\lambda-\epsilon)|^2}{2\epsilon}\,d\lambda$$

and

$$(11) \qquad d\Psi(t:f) = w^{*}\text{-}\lim_{\epsilon\downarrow 0}\frac{|g(\lambda+\epsilon)-g(\lambda-\epsilon)|^2}{2\epsilon}\,d\lambda.$$

Theorem 3. Let $x(t)$ be a strictly stationary process with mean 0 and continuous covariance function $r(t)$ $(=\int e^{i\lambda t}\,dF(\lambda))$. Then the regularization $f(t,\omega)$ of $x(t)$ (as a function of t) belongs to $\mathcal{W}$ for a.a. ω.

Furthermore, if $x(t, w)$ is ergodic, then

(12) $$R(t : f(\cdot, w)) = r(t) \qquad \text{for a.a.w.}$$

and so

(13) $$d\Psi(\lambda : f(\cdot, w)) = dF(\lambda) \qquad \text{for a.a.w.}$$

Proof. Consider $y(w) = f(t, w)\overline{f(0, w)}$ for any fixed t. Then y is $\mathcal{B}(x)$-measurable, $y \in L^1(\Omega, \mathcal{B}(x), P)$ and

$$\Phi_s y(w) = f(t + s, w)\overline{f(s, w)}$$

(see page IV 5.6 for the shift operator Φ_s). The individual ergodic theorem shows that

$$R(t, w) \equiv R(t, f(\cdot, w)) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t+s, w)\overline{f(s, w)}\, ds$$

exists for a.a.w., belongs to $L^1(\Omega, \mathcal{B}(x), P)$, is invariant under the shift Φ_s and

$$E[R(t, w)] = E(y) = r(t)$$

Since $f(t, w)$ is Borel measurable in (t, w), we can use Fubini's theorem to conclude that, for a.a.w. $R(t, w)$ is determined for a.a.t. Since $R(0, w)$ is determined for a.a.w, we can say that, for $w \in \Omega_1$, $P(\Omega_1) = 1$, $R(t, w)$ is determined for a.a.t including $t = 0$. By the remark mentioned above, $|R(t, w)| \leq |R(0, w)|$,

$$\tilde{R}(w) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} R(t, w)\, dt$$

exists and is real and $\leq R(0, w)$ for $w \in \Omega_1$. But

$$\begin{aligned} &E[R(0, w) - \tilde{R}(w)] \\ &= r(0) - E[\tilde{R}(w)] \\ &= r(0) - \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} E[R(t, w)]\, dt \\ &= r(0) - \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} r(t)\, dt = 0 \end{aligned}$$

Therefore $\tilde{R} = R(0, w)$ for $w \in \Omega_2$, $P(\Omega_2) = 1$. Thus (1), (2), (3) hold for $f(t, w)$ if $w \in \Omega_1 \cap \Omega_2$, which proves the first half of our theorem.

To prove the second half, notice that $R(t, w)$ is invariant under shifts and so independent of w.

<u>Remark 2</u>. If we apply Theorem 2 to $f(t, w)$ in Theorem 2, we can get

$$f(t, w) = \lim_{\epsilon \downarrow 0} \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} e^{it\lambda} \frac{g(\lambda + \epsilon) - g(\lambda - \epsilon)}{2\epsilon} d\lambda \qquad \text{for a.a.} w$$

(Notice l.i.m. means limit in the $L^2((-\infty, \infty), dt)$ norm) and this is a concrete version of $f = \mathcal{F}(DG)$ established on page VIII.8 (Theorem 5); It is easy to see that $g(\lambda, w) - G(\lambda, w)$ is a constant (depending only on w) for a.a.w.

<u>Theorem 4</u>. Given any bounded measure dF there exists at least one function whose spectral measure is dF.

<u>Proof</u>. In page III.25 (Theorem 5) we proved the existence of an ergodic strictly stationary process with the covariance function $r(t) = \int e^{i\lambda t} dF(\lambda)$. Then the second part of Theorem 3 proved above shows that almost all sample functions (regularization) has the auto-correlation function $r(t)$ and so the spectral measure dF.