V. Entropy

1. Introduction.

All Kolmogorov automorphisms are σ-Lebesgues and therefore unitary equivalent to each other.

QUESTION Are they all measure-isomorphic ?

The answer is NO according to Sinai.

Let $\{x_n\}$ be a stationary independent sequence with $P(x_n = k) = p_k$, $k = 0,1,\dots,N$ and $S = S(p) : \mathcal{B}(x) \to \mathcal{B}(x)$ be the associated automorphism(shift). Then S is always a Kolmogorov automorphism; $\mathcal{B}(x_0, x_{-1}, x_{-2}, \dots)$ plays the role of $\mathcal{B}_0$ in the Kolmogorov-Sinai theorem. But $\sum p_i \log p_i = \sum q_j \log q_j$ is necessary for $S(p) \cong S(q)$. To see this we shall introduce the entropy $h(S)$ of S and prove that

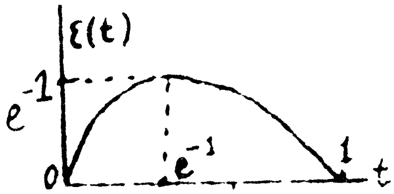
1. $S \cong T \implies h(S) = h(T)$
2. $h(S(p)) = -\sum p_i \log p_i$

2. Preliminary Notations.

(a)

$\Omega(\mathcal{B}_\Omega, P)$	the basic probability measure space
S, T	automorphism : $\mathcal{B}_\Omega \to \mathcal{B}_\Omega$
$\mathcal{A}, \mathcal{B}$	finite subalgebras of $\mathcal{B}_\Omega$
$A_i, i = 1,2,\dots,m$	atoms of $\mathcal{A}$
$B_j, j = 1,2,\dots,n$	atoms of $\mathcal{B}$
$\mathcal{C}, \mathcal{D}$	arbitrary Borel subalgebras of $\mathcal{B}_\Omega$
$\mathcal{N}$	trivial subalgebra of $\mathcal{B}_\Omega$

(b) $$\varepsilon(t) = \begin{cases} -t \log t & (0 < t \leq 1) \\ 0 & (t = 0) \end{cases}$$



(i) $\varepsilon(t)$ is continuous, concave and $\leq \varepsilon(t) \leq t$ $\varepsilon(0) = \varepsilon(1) = 0$

(ii) subadditive: $\varepsilon(\sum t_i) \leq \sum \varepsilon(t_i)$.

(iii) $0 \leq \xi(\omega) \leq 1 \Rightarrow \varepsilon[E(\xi|\mathcal{C})] \geq E(\varepsilon(\xi)|\mathcal{C})$.

2. <u>The entropy of finite algebras.</u>

entropy $H(\mathcal{A}) = \sum_{i=1}^{m} \varepsilon(P(A_i))$ $\left(\in [0, m e^{-1}]\right)$

conditional entropy $H(\mathcal{A}|\mathcal{C}) = \sum_i \varepsilon(P(A_i|\mathcal{C}))$;

mean conditional entropy $\bar{H}(\mathcal{A}|\mathcal{C}) = E(H(\mathcal{A}|\mathcal{C}))$.

(a) $H(\mathcal{A}) = \bar{H}(\mathcal{A}|\mathcal{N})$

$\bar{H}(\mathcal{A}|\mathcal{C}) = 0$ if $\mathcal{C} \supset \mathcal{A}$

$\bar{H}(\mathcal{A}|\mathcal{B}) = H(\mathcal{A})$ if $\mathcal{A}$ and $\mathcal{B}$ are independent.

(b) (monotone)

$\mathcal{A} \subset \mathcal{B} \Rightarrow \bar{H}(\mathcal{A}|\mathcal{C}) \leq \bar{H}(\mathcal{B}|\mathcal{C})$

$\mathcal{C} \subset \mathcal{D} \Rightarrow \bar{H}(\mathcal{A}|\mathcal{C}) \geq \bar{H}(\mathcal{A}|\mathcal{D})$

<u>Corollary</u> $0 \leq \bar{H}(\mathcal{A}|\mathcal{C}) \leq H(\mathcal{A})$

(c) (additive)

$$H(\mathcal{A} \vee \mathcal{B}) = H(\mathcal{A}) + \bar{H}(\mathcal{B}|\mathcal{A})$$

$$H(\mathcal{A} \vee \mathcal{B}) = H(\mathcal{A}) + H(\mathcal{B}) \quad \text{if } \mathcal{A} \text{ and } \mathcal{B} \text{ are indep.}$$

(d) (subadditive)

$$H(\mathcal{A} \vee \mathcal{B}) \leq H(\mathcal{A}) + H(\mathcal{B})$$

$$\bar{H}(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) \leq \bar{H}(\mathcal{A}|\mathcal{C}) + \bar{H}(\mathcal{B}|\mathcal{C})$$

(e) (continuous) $\mathcal{C}_k \uparrow \mathcal{C} \Rightarrow \bar{H}(\mathcal{A}|\mathcal{C}_k) \downarrow \bar{H}(\mathcal{A}|\mathcal{C})$

(f) (invariant) $\bar{H}(S\mathcal{A}|S\mathcal{C}) = \bar{H}(\mathcal{A}|\mathcal{C})$, $H(S\mathcal{A}) = H(\mathcal{A})$

3. <u>The entropy of automorphism.</u>

(a) <u>Lemma.</u> $f(n) = H(\bigvee_{i=0}^{n-1} S^{-i}\mathfrak{a})$ is subadditive in n and so $\lim_n f(n)/n$ exists and in $[0, f(1)]$.

(b) <u>Definition</u>

entropy of S with respect to $\mathfrak{a}$:

$$h(S,\mathfrak{a}) = \lim_n H(\bigvee^{n-1} S^{-i}\mathfrak{a})/n \quad (\in [0, H(\mathfrak{a})])$$

entropy of S :

$$h(S) = \sup_{\mathfrak{a}} h(S,\mathfrak{a})$$

<u>Corollary.</u> $S \cong T \Rightarrow h(S) = h(T)$.

(c) <u>Theorem</u> (strongly monotone)

$$\mathfrak{a} \subset \bigvee_i S^i \mathfrak{B} \Rightarrow \cancel{H(\mathfrak{a}) \leq H(\mathfrak{B})}\ h(S,\mathfrak{a}) \leq h(S,\mathfrak{B})$$

<u>Corollary</u> $h(S) = h(S,\mathfrak{a})$ if $\bigvee_i S^i \mathfrak{a} = \mathfrak{B}_\Omega$

(d) <u>Example</u> $h(S(p)) = -\sum_i p_i \log p_i$.